Robust Fault Detection and Set-theoretic UIO for Discrete-time LPV Systems with State and Output Equations Scheduled by Inexact Scheduling Variables

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***Abstract*—This paper proposes a novel robust fault detection (FD) approach and designs a set-theoretic unknown input observer (SUIO) for linear parameter-varying (LPV) systems with both state and output equations scheduled by inexact scheduling variables. First, for such LPV systems, we propose a novel robust FD method by combing the set theory with the unknown input observer (UIO), which considers the bounds of measurement errors of scheduling variables to generate FD-oriented sets. In general, as long as sensors with sufficiently high precision are equipped to measure the scheduling variables, the bounds of measurement errors of scheduling variables can be less conservative than those direct bounds of scheduling variables, which can reduce robust FD conservatism in this way. Second, we give the unknown input decoupling condition of SUIO for such LPV systems and propose an SUIO design method under this condition for robust state estimation (SE). Besides, stability conditions for the proposed methods are established via matrix inequalities. At the end of this paper, a case study is used to illustrate the effectiveness of the proposed methods.**

***Index Terms*—Fault detection, state estimation, unknown input observer, set theory, LPV systems, inexact scheduling variables.**

I. Introduction

The unknown input observer (UIO) plays an important role in robust state estimation (SE) and fault detection (FD). If there exist UIOs for a system, we can design UIOs to remove the effect of its unknown inputs on SE and FD. In the literature, the design of UIOs was studied for different types of systems, such as linear time-invariant (LTI) systems [7], [17], linear time-delay systems [38], [39], linear switched systems [2], [22], linear parameter-varying systems (LPV) [18] and so on.

In the literature, most results on UIOs are for linear systems, which are generally divided into three types. The first type is based on the classical design condition in [20] for the complete decoupling of unknown inputs, see [9], [17] and among others. The second one aims to avoid the classical design condition under some assumptions on unknown inputs, see [7], [26] and among others. However, both ways cannot overcome the limits of the UIO design condition. To solve this problem, the works in [14], [34] proposed to divide unknown inputs into two groups. If partial unknown inputs satisfy their corresponding decoupling condition, a partially-decoupled UIO can be constructed because the number of unknown inputs chosen to be decoupled is adjustable to satisfy their decoupling condition.

Some interesting results on UIO design, SE and FD problems for LPV systems have been recently achieved. In [18], under the assumption that the LPV system is uniformly strongly algebraically observable with respect to its scheduling variables, a design approach of UIO for LPV systems with linear time-invariant output equation was proposed. In [16], a UIO was designed for the LPV descriptor system with time delays and an actuator fault detection and isolation (FDI) approach was further developed for the system. The works proposed in [25], [28] applied the set-based methods (interval observers and set-valued observers) into robust FD of LPV systems, where the bounds of scheduling variables are used to generate adaptive thresholds of residuals for robustness. In [6], the authors proposed an FD approach for LPV systems with imperfect scheduling parameters by using a sliding mode observer, which only considers the LPV system with a constant output matrix. The work in [29] combining the interval observer and the UIO was proposed to estimate the upper and lower bounds of states and implement robust fault diagnosis for an uncertain LPV system with parametric uncertainties and a constant output matrix. In order to improve FD robustness and attenuate unknown inputs, an FD method based on the H_- index and H_∞ norm was proposed in [13] (some methods using similar techniques can be found in [8], [12], [32], [33] and among others). More works for FD of LPV systems can be found in [3], [27] and among others.

It is worth mentioning that most of works on UIO design and robust FD only consider the LPV system with a constant output matrix and perfectly known scheduling variables, while a more general situation is that both state and output equations have the LPV form and/or the scheduling variables are corrupted by measurement errors from sensors. From our best knowledge, there exist two works considering UIO design for the LPV system with an LPV output matrix. The first work was proposed in [19], which was an extension of [18]. The second work was proposed in [31], which implemented robust state and fault estimation for the T-S system (similar to the LPV system). Particularly, the work in [31] first used the system state equation to solve an expression of a fault f_k and then substituted the expression of f_k back into the system state equation to obtain an equivalent system without explicitly including f_k (i.e., decouple f_k) for fault estimation. This proposed idea can also be used to decouple unknown inputs

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(see [31] for details). However, both works did not consider the situation that the scheduling variables are corrupted by the measurement errors originated from sensors.

Different from the existing works in the literature, this paper focuses on robust FD and set-theoretic unknown input observer (SUIO) design for LPV systems with both state and output equations scheduled by inexact scheduling variables, where the set theory deals with the measurement errors of scheduling variables instead of passively decoupling part of unknown inputs and the notion of UIO is used to remove the effect of unknown inputs on the known part of system dynamics instead of completely decoupling part of unknown inputs. The SUIO was first proposed by the authors in [34], [37]. In [37], the set theory and the UIO were combined to implement robustness of FD and SE of LPV systems with an LTI output equation and perfectly measurable scheduling variables by dividing the unknown inputs into two groups, where the first group is actively decoupled by the UIO and the second group is passively decoupled by the set theory.

The contributions of this paper are summarized as follows:

- in Section III, for the class of LPV systems with both state and output equations scheduled by inexact scheduling variables, this paper proposes a novel robust FD method by dividing the system dynamics into a known part and an unknown part, then employing the notion of UIO to remove the effect of unknown inputs and/or noises on the known part, and eventually using the set theory to handle the unknown part to achieve FD robustness;
- in Section IV, for the class of LPV systems with a constant output matrix and inexact scheduling variables, a novel SUIO design method is proposed for robust SE, which is based on a new UIO framework composed of an augmented dynamics integrating the traditional UIO and its SE error dynamics for more design degrees of freedom of SUIO parametric matrices.

In this paper, Section II introduces the general LPV and UIO models. Section III proposes a novel robust FD method for the general LPV systems. Section IV first gives the unknown input decoupling condition of UIO for LPV systems and then proposes an SUIO design method with more degrees of freedom for the LPV systems. Finally, the effectiveness of the proposed methods is illustrated in Section V and the paper is finally concluded in Section VI.

II. Problem Formulation

This section introduces an LPV system with both state and output equations scheduled by inexact scheduling variables and presents the corresponding UIO form for this LPV system.

A. System Description

This paper considers the class of discrete-time nonlinear systems modelled by the following discrete-time LPV model:

$$x_{k+1} = A(\rho_k)x_k + B(\rho_k)u_k + E(\rho_k)\omega_k, \tag{1a}$$
$$y_k = C(\rho_k)x_k + F(\rho_k)\eta_k, \tag{1b}$$

where $A(\rho_k) \in \mathbb{R}^{n\times n}$, $B(\rho_k) \in \mathbb{R}^{n\times p}$, $E(\rho_k) \in \mathbb{R}^{n\times r}$, $C(\rho_k) \in \mathbb{R}^{q\times n}$ and $F(\rho_k) \in \mathbb{R}^{q\times s}$ are parametric matrices dependent of a scheduling vector $\rho_k \in \mathbb{R}^m$, $k \in \mathbb{N}$ denotes the time instant, $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^q$ denote the state and output vectors, $u_k \in \mathbb{R}^p$ and $\omega_k \in \mathbb{R}^r$ represent the known and unknown inputs (including disturbances, modeling errors, etc.), and $\eta_k \in \mathbb{R}^s$ represents the measurement noise vector.

It is assumed that ρ_k is bounded by a hypercube $\mathcal{P}$, i.e., $\rho_k \in \mathcal{P}$, where ρ_k is with the form $[\rho_k^1, \rho_k^2, \cdots, \rho_k^m]^T$ (ρ_k^i is the i-th component). Thus, a parameterized matrix $J(\rho_k)$ (an affine function of ρ_k) is also bounded by a polytopic set and can be represented as a linear combination of vertex matrices of this set (i.e., polytopic decomposition of $J(\rho_k)$) as

$$J(\rho_k) = \sum_{i=1}^{N} \lambda_i(\rho_k)J_i, \tag{2}$$

where J_i is the i-th vertex matrix of the set of $J(\rho_k)$ (J can represent A, B, C, E and F in (1)), $N = 2^m$ is the number of vertex matrices and $\lambda_i(\rho_k)$ is the weighting coefficient of the i-th vertex matrix with $\sum_{i=1}^{N} \lambda_i(\rho_k) = 1$, $0 \le \lambda_i(\rho_k) \le 1$.

In some existing works on the LPV systems, the scheduling vector ρ_k is often considered being perfectly measurable. However, in reality, there always exists a measurement error between the actual value ρ_k and a measured value $\hat{\rho}_k$ of ρ_k. Here we consider this general case, which is denoted as

$$\hat{\rho}_k = \rho_k + \tilde{\rho}_k, \tag{3}$$

where $\tilde{\rho}_k$ denotes the measurement error vector of sensors equipped in the system to measure the scheduling variables.

The measurement errors of sensors are originated from their measurement precision and it is considered that the measurement precision of sensors used to obtain the bounds of measurement errors can be read from their performance instructions, which are denoted as $\tilde{\rho}_k \in \tilde{\mathcal{P}}$ where $\tilde{\mathcal{P}} = \begin{bmatrix}\tilde{\mathcal{P}}_1 & \tilde{\mathcal{P}}_2 & \dots & \tilde{\mathcal{P}}_m\end{bmatrix}^T$ is the bounding hypercube of $\tilde{\rho}_k$ and

$$\tilde{\rho}_k^i \in \tilde{\mathcal{P}}_i = [\underline{\tilde{\rho}}_{i,k}, \ \overline{\tilde{\rho}}_{i,k}], \tag{4}$$

where $\tilde{\mathcal{P}}_i$ is the i-th interval component of $\tilde{\mathcal{P}}$, $\tilde{\rho}_k^i = \hat{\rho}_k^i - \rho_k^i$ is the i-th component of $\tilde{\rho}_k$, and $\underline{\tilde{\rho}}_{i,k}$ and $\overline{\tilde{\rho}}_{i,k}$ are the lower and upper bounds of $\tilde{\rho}_k^i$, respectively.

B. UIO Structure

According to [18], [34], a general UIO form for the LPV system (1) is given as

$$z_{k+1} = N(\hat{\rho}_k)z_k + T(\hat{\rho}_k)u_k + K(\hat{\rho}_k)y_k, \tag{5a}$$
$$\hat{x}_k = z_k + H(\hat{\rho}_k)y_k, \tag{5b}$$

where $N(\hat{\rho}_k)$, $T(\hat{\rho}_k)$, $K(\hat{\rho}_k)$ and $H(\hat{\rho}_k)$ are compatible parametric matrices, z_k is the state vector of the UIO, $\hat{x}_k$ is the estimated state vector of (1) and $\hat{\rho}_k = \left[\hat{\rho}_k^1, \hat{\rho}_k^2, \cdots, \hat{\rho}_k^m\right]^T$ denotes an actual measurement of ρ_k with

$$\hat{\rho}_k \in \mathcal{P}, \tag{6}$$

where $\hat{\rho}_k$ is different from ρ_k and $\hat{\rho}_k^i$ denotes the i-th component of $\hat{\rho}_k$. Note that, by projecting $\hat{\rho}_k$ into the bounding set

$\mathcal{P}$, we can always obtain the measurement $\hat{\rho}_k$ confined inside $\mathcal{P}$ (see more explanations on this point in [24]). Thus, we can also have the polytopic decomposition

$$J(\hat{\rho}_k) = \sum_{i=1}^{N} \lambda_i(\hat{\rho}_k) J_i, \tag{7}$$

where J can represent A, B, C, E, F, N, T, K and H, and $\lambda_i(\hat{\rho}_k)$ has the similar definition with $\lambda_i(\rho_k)$ as $\sum_{i=1}^{N} \lambda_i(\hat{\rho}_k) = 1$, $0 \leq \lambda_i(\hat{\rho}_k) \leq 1$.

Assumption 1. *$J(\rho_k)$ (or $J(\hat{\rho}_k)$) is the affine function of ρ_k (or $\hat{\rho}_k$), i.e., $J(\rho_k) = J^0 + \sum_{i=1}^{m} J^i \rho_k^i$ (or $J(\hat{\rho}_k) = J^0 + \sum_{i=1}^{m} J^i \hat{\rho}_k^i$), where J^0, J^1, $\cdots$, and J^m are constant matrices.*

Under Assumption 1 and using (3), we could have

$$J(\hat{\rho}_k) = J(\rho_k) + \sum_{i=1}^{m} J^i \tilde{\rho}_k^i. \tag{8}$$

The meanings of J_i in (7) and J^i in (8) are different. J_i denotes the i-th vertex matrix of the hypercube of $J(\rho_k)$ (or $J(\hat{\rho}_k)$), while J^i denotes the coefficient matrix of the i-th component of ρ_k (or $\hat{\rho}_k$) of linear form of $J(\rho_k)$ (or $J(\hat{\rho}_k)$).

Remark 1. *For a clear explanation of the proposed methods, both the affine and polytopic forms of LPV systems are used in the mathematical derivations of this paper. However, these two forms of LPV systems are actually equivalent to each other.*

C. Analysis of System Behaviors

Using (1) and (5), we define the SE error as

$$e_k = x_k - \hat{x}_k, \tag{9}$$

whose dynamics can be further derived as

$$\begin{aligned} e_{k+1} =& \Lambda_0 e_k + \Lambda_1 z_k + \Lambda_2 y_k + \Lambda_3 u_k + \Lambda_4 \omega_k \\ & + \Lambda_5 \eta_{k+1} + \Lambda_6 \eta_k \end{aligned} \tag{10}$$

with

$$\Lambda_0 = A_k^{\rho} - H_{k+1}^{\hat{\rho}} C_{k+1}^{\rho} A_k^{\rho} - K_{1,k}^{\hat{\rho}} C_k^{\rho}, \tag{11a}$$
$$\Lambda_1 = A_k^{\rho} - H_{k+1}^{\hat{\rho}} C_{k+1}^{\rho} A_k^{\rho} - K_{1,k}^{\hat{\rho}} C_k^{\rho} - N_k^{\hat{\rho}}, \tag{11b}$$
$$\Lambda_2 = (A_k^{\rho} - H_{k+1}^{\hat{\rho}} C_{k+1}^{\rho} A_k^{\rho} - K_{1,k}^{\hat{\rho}} C_k^{\rho}) H_k^{\hat{\rho}} - K_{2,k}^{\hat{\rho}}, \tag{11c}$$
$$\Lambda_3 = B_k^{\rho} - T_k^{\hat{\rho}} - H_{k+1}^{\hat{\rho}} C_{k+1}^{\rho} B_k^{\rho}, \tag{11d}$$
$$\Lambda_4 = E_k^{\rho} - H_{k+1}^{\hat{\rho}} C_{k+1}^{\rho} E_k^{\rho}, \tag{11e}$$
$$\Lambda_5 = - H_{k+1}^{\hat{\rho}} F_{k+1}^{\rho}, \tag{11f}$$
$$\Lambda_6 = - K_{1,k}^{\hat{\rho}} F_k^{\rho} \tag{11g}$$

with $K_k^{\hat{\rho}} = K_{1,k}^{\hat{\rho}} + K_{2,k}^{\hat{\rho}}$, where we simplify the notations $J(\rho_k)$, $J(\hat{\rho}_k)$, $\lambda_i(\rho_k)$, and $\lambda_i(\hat{\rho}_k)$ as J_k^{ρ}, $J_k^{\hat{\rho}}$, $\lambda_{i,k}^{\rho}$ and $\lambda_{i,k}^{\hat{\rho}}$, respectively. By substituting (8) into (11), we further have

$$\Lambda_0 = \hat{\Lambda}_0 + \tilde{\Lambda}_0, \tag{12a}$$
$$\Lambda_1 = \hat{\Lambda}_1 + \tilde{\Lambda}_1, \tag{12b}$$
$$\Lambda_2 = \hat{\Lambda}_2 + \tilde{\Lambda}_2, \tag{12c}$$
$$\Lambda_3 = \hat{\Lambda}_3 + \tilde{\Lambda}_3, \tag{12d}$$
$$\Lambda_4 = \hat{\Lambda}_4 + \tilde{\Lambda}_4, \tag{12e}$$
$$\Lambda_5 = \hat{\Lambda}_5 + \tilde{\Lambda}_5, \tag{12f}$$
$$\Lambda_6 = \hat{\Lambda}_6 + \tilde{\Lambda}_6 \tag{12g}$$

with

$$\hat{\Lambda}_0 = A_k^{\hat{\rho}} - H_{k+1}^{\hat{\rho}} C_{k+1}^{\hat{\rho}} A_k^{\hat{\rho}} - K_{1,k}^{\hat{\rho}} C_k^{\hat{\rho}}, \tag{13a}$$
$$\hat{\Lambda}_1 = A_k^{\hat{\rho}} - H_{k+1}^{\hat{\rho}} C_{k+1}^{\hat{\rho}} A_k^{\hat{\rho}} - K_{1,k}^{\hat{\rho}} C_k^{\hat{\rho}} - N_k^{\hat{\rho}}, \tag{13b}$$
$$\hat{\Lambda}_2 = (A_k^{\hat{\rho}} - H_{k+1}^{\hat{\rho}} C_{k+1}^{\hat{\rho}} A_k^{\hat{\rho}} - K_{1,k}^{\hat{\rho}} C_k^{\hat{\rho}}) H_k^{\hat{\rho}} - K_{2,k}^{\hat{\rho}}, \tag{13c}$$
$$\hat{\Lambda}_3 = B_k^{\hat{\rho}} - T_k^{\hat{\rho}} - H_{k+1}^{\hat{\rho}} C_{k+1}^{\hat{\rho}} B_k^{\hat{\rho}}, \tag{13d}$$
$$\hat{\Lambda}_4 = E_k^{\hat{\rho}} - H_{k+1}^{\hat{\rho}} C_{k+1}^{\hat{\rho}} E_k^{\hat{\rho}}, \tag{13e}$$
$$\hat{\Lambda}_5 = - H_{k+1}^{\hat{\rho}} F_{k+1}^{\hat{\rho}}, \tag{13f}$$
$$\hat{\Lambda}_6 = - K_{1,k}^{\hat{\rho}} F_k^{\hat{\rho}} \tag{13g}$$

and

$$\begin{aligned} \tilde{\Lambda}_0 =& -\sum_{i=1}^{m} A^i \tilde{\rho}_k^i - H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i)(\sum_{i=1}^{m} A^i \tilde{\rho}_k^i) \\ & + H_{k+1}^{\hat{\rho}} C_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} A^i \tilde{\rho}_k^i) + H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i) A_k^{\hat{\rho}} \\ & + K_{1,k}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_k^i), \end{aligned} \tag{14a}$$

$$\begin{aligned} \tilde{\Lambda}_1 =& -(\sum_{i=1}^{m} A^i \tilde{\rho}_k^i) - H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i)(\sum_{i=1}^{m} A^i \tilde{\rho}_k^i) \\ & + H_{k+1}^{\hat{\rho}} C_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} A^i \tilde{\rho}_k^i) + H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i) A_k^{\hat{\rho}} \\ & + K_{1,k}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i), \end{aligned} \tag{14b}$$

$$\begin{aligned} \tilde{\Lambda}_2 =& -(\sum_{i=1}^{m} A^i \tilde{\rho}_k^i) H_k^{\hat{\rho}} - H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i)(\sum_{i=1}^{m} A^i \tilde{\rho}_k^i) H_k^{\hat{\rho}} \\ & + H_{k+1}^{\hat{\rho}} C_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} A^i \tilde{\rho}_k^i) H_k^{\hat{\rho}} + K_{1,k}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i) H_k^{\hat{\rho}} \\ & + H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i) A_k^{\hat{\rho}} H_k^{\hat{\rho}}, \end{aligned} \tag{14c}$$

$$\begin{aligned} \tilde{\Lambda}_3 =& H_{k+1}^{\hat{\rho}} C_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} B^i \tilde{\rho}_k^i) - H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i)(\sum_{i=1}^{m} B^i \tilde{\rho}_k^i) \\ & - (\sum_{i=1}^{m} B^i \tilde{\rho}_k^i) + H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i) B_k^{\hat{\rho}}, \end{aligned} \tag{14d}$$

$$\begin{aligned} \tilde{\Lambda}_4 =& H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i) E_k^{\hat{\rho}} - H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i)(\sum_{i=1}^{m} E^i \tilde{\rho}_k^i) \\ & - (\sum_{i=1}^{m} E^i \tilde{\rho}_k^i) + H_{k+1}^{\hat{\rho}} C_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} E^i \tilde{\rho}_k^i), \end{aligned} \tag{14e}$$

$$\tilde{\Lambda}_5 = H_{k+1}^{\hat{\rho}} (\sum_{i=1}^{m} F^i \tilde{\rho}_{k+1}^i) = \sum_{i=1}^{m} (H_{k+1}^{\hat{\rho}} F^i) \tilde{\rho}_{k+1}^i, \tag{14f}$$

$$\tilde{\Lambda}_6 = K_{1,k}^{\hat{\rho}} (\sum_{i=1}^{m} F^i \tilde{\rho}_k^i) = \sum_{i=1}^{m} (K_{1,k}^{\hat{\rho}} F^i) \tilde{\rho}_k^i. \tag{14g}$$

Since ρ_k is unknown in the LPV system (1), it is impossible to exactly know Λ_0, Λ_1, Λ_2, Λ_3, Λ_4, Λ_5 and Λ_6 in (11). This means that it is difficult to directly design Λ_1, Λ_2, Λ_3 and Λ_4 to be zeros to apply the classical idea of the traditional UIO to remove the effect of unknown inputs. However, as seen in

(12), (13) and (14), the SE error dynamics can be divided into a known part described by $\hat{\Lambda}_0$, $\hat{\Lambda}_1$, $\hat{\Lambda}_2$, $\hat{\Lambda}_3$, $\hat{\Lambda}_4$, $\hat{\Lambda}_5$ and $\hat{\Lambda}_6$, and an unknown part described by $\tilde{\Lambda}_0$, $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$, $\tilde{\Lambda}_3$, $\tilde{\Lambda}_4$, $\tilde{\Lambda}_5$ and $\tilde{\Lambda}_6$. Here, we can consider designing the parametric matrices to remove the effect of unknown inputs on the known part. But for the unknown part of the dynamics, although the exact values of its parametric matrices cannot be obtained, we can still use the set theory to compute the sets of the corresponding signals by considering the bounds of $\tilde{\rho}_k$. Based on these ideas above, it is still possible to use the set theory and the notion of UIO to implement robust FD and SE for the LPV system (1). Moreover, the effect of unknown inputs on the known part of dynamics is removed and consequently the conservatism of robust FD and SE can be reduced to some extent.

This paragraph briefly introduces Sections III and IV to help the readers understand the logic of this paper. On one hand, for the LPV systems (1), we propose a novel robust FD method for such LPV systems based on the set theory and the notion of UIO. The proposed robust FD method can be applied to the LPV systems (1) with both state and output equations scheduled by inexact scheduling variables and is free from the limit of Assumption 2 given in Section IV. This part of contributions will be presented in Section III. On the other hand, under Assumption 2 (i.e., the unknown input decoupling condition), a novel SUIO is proposed for a reduced version (46) of the LPV systems (1). The proposed SUIO can only be designed for the reduced LPV system (46) with a constant output matrix (i.e., the LPV system (1) under Assumption 2). This part of contributions will be presented in Section IV.

III. Robust Fault Detection of LPV Systems

This section proposes a novel robust FD method for the LPV systems (1) with state and output equations scheduled by inexact scheduling variables. In general, due to the measurement errors in the scheduling variables, we cannot actively decouple unknown inputs. However, we still use the notion of UIO to remove the effect of unknown inputs on the known part (13) of the SE error dynamics and then further combine the set theory to handle the remaining uncertainties to achieve FD robustness and simultaneously reduce the FD conservatism.

A. Analysis of FD Conservatism

For the LPV systems, a general case is that both E_k^{ρ} and C_k^{ρ} are dependent of ρ_k and unknown as shown in (1). By analyzing (10), we see that ω_k, η_k and η_{k+1} are unknown while z_k, y_k and u_k are available. Based on the general idea of set-based FD methods, we have to consider the bounds of ω_k, η_k and η_{k+1} instead of themselves to generate residual sets for robust FD. Thus, in order to reduce the FD conservatism, the idea is to reduce the effect of ω_k, η_k and η_{k+1} on the sizes of residual sets as much as possible. Contrarily, since z_k, y_k and u_k are known, they do not affect the sizes of residual sets. Thus, from the FD conservatism viewpoint, we should reduce the effects of the terms related to ω_k, η_k and η_{k+1} in (10) (i.e., the terms $\Lambda_4\omega_k$, $\Lambda_5\eta_{k+1}$ and $\Lambda_6\eta_k$ in (10)).

According to (11), since the parametric matrices Λ_4, Λ_5 and Λ_6 of the LPV system (1) are unknown, it is impossible to design the parametric matrices of (5) to completely remove the effect of ω_k, η_k and η_{k+1} (i.e., impossible to make all Λ_4, Λ_5 and Λ_6 be zeros in (10)). But, we can try to design the parametric matrices of (5) to reduce the effect of ω_k, η_k and η_{k+1} as much as possible. As done in (13) and (14), the parametric matrices of (10) can be divided into a known part (13) and an unknown part (14). Even though it is impossible to eliminate the unknown part, it is still possible to remove some terms of the known part by turning to the notion of UIO, which is the main idea to reduce the FD conservatism here.

We propose to design proper parametric matrices $N_k^{\hat{\rho}}$, $T_k^{\hat{\rho}}$, $K_k^{\hat{\rho}}$ and $H_k^{\hat{\rho}}$ to eliminate part of effects of ω_k, η_k and η_{k+1} in (10). Similarly, since z_k, y_k and u_k are known, it is unnecessary to design $\hat{\Lambda}_1$, $\hat{\Lambda}_2$, $\hat{\Lambda}_3$ to be zeros in (10). But ω_k, η_k and η_{k+1} are unknown, when using the set theory to handle them for robust FD, we have to consider and propagate their bounds through the dynamics online to obtain residual sets. Thus, in order to reduce the FD conservatism, we should reduce the effect of ω_k, η_k and η_{k+1} as much as possible.

By analyzing (13e), (13f) and (13g), we can know that in some sense, there is no solution simultaneously making (13e) and (13f) be zeros. But it is possible to simultaneously design (13e) and (13g) (or ((13f) and (13g))) be zeros. Thus, we can have two general situations on reducing the FD conservatism originated from ω_k, η_k and η_{k+1}:

- Situation 1: design both (13e) and (13g) to be zeros;
- Situation 2: design both (13f) and (13g) to be zeros.

From practical viewpoints, we can compare these two situations and select a better one to design a robust FD algorithm by considering the system features to obtain a better FD performance (see [35] for a method that may be used to compare the FD conservatism of these two situations).

B. The First Situation

When Situation 1 is chosen, we design $H_k^{\hat{\rho}}$ and $K_{1,k}^{\hat{\rho}}$ such that $\hat{\Lambda}_4 = \mathbf{0}$ and $\hat{\Lambda}_6 = \mathbf{0}$ (see (13e) and (13g)). Similarly, according to [10], the condition to solve $\hat{\Lambda}_4 = \mathbf{0}$ is

$$\operatorname{rank}(E_k^{\hat{\rho}}) = \operatorname{rank}(C_{k+1}^{\hat{\rho}} E_k^{\hat{\rho}}) \text{ for all } \hat{\rho}_k \in \mathcal{P}. \tag{15}$$

Remark 2. *The verification of the condition (15) consists in checking the matrix ranks of*

$$E_k^{\hat{\rho}} = \sum_{i=1}^{N} \lambda_{i,k}^{\hat{\rho}} E_i, \tag{16a}$$

$$C_{k+1}^{\hat{\rho}} E_k^{\hat{\rho}} = \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_{i,k}^{\hat{\rho}} \lambda_{j,k+1}^{\hat{\rho}} C_j E_i, \tag{16b}$$

which can be done by using some results shown in [31].

Under the condition (15), we solve (13e) to obtain

$$\begin{aligned} H_{k+1}^{\hat{\rho}} = {} & E_k^{\hat{\rho}}[(C_{k+1}^{\hat{\rho}} E_k^{\hat{\rho}})^T C_{k+1}^{\hat{\rho}} E_k^{\hat{\rho}}]^{-1} (C_{k+1}^{\hat{\rho}} E_k^{\hat{\rho}})^T \\ & + \mathring{H}\Big\{ I - C_{k+1}^{\hat{\rho}} E_k^{\hat{\rho}} [(C_{k+1}^{\hat{\rho}} E_k^{\hat{\rho}})^T C_{k+1}^{\hat{\rho}} E_k^{\hat{\rho}}]^{-1} \\ & (C_{k+1}^{\hat{\rho}} E_k^{\hat{\rho}})^T \Big\}, \end{aligned} \tag{17}$$

where $(\cdot)^{-1}$ and $(\cdot)^T$ denote the pseudo inverse and transpose, respectively and $\mathring{H}$ is an arbitrary compatible matrix.

Lemma 1. *Given known compatible matrices Q and R, if $RQ^{\{1\}}Q = R$ holds, then there exists a solution X such that $XQ = R$. The general solution is obtained as $X = RQ^{\{1\}} + Y(I - QQ^{\{1\}})$, where Y is an arbitrary compatible matrix, I is the compatible identity matrix and $Q^{\{1\}}$ is the type-1 generalized inverse of the matrix Q.*

By using Lemma 1, the solution of $\hat{\Lambda}_6 = \mathbf{0}$ is

$$K_{1,k}^{\hat{\rho}} = Y_1(I - F_k^{\hat{\rho}}(F_k^{\hat{\rho}})^{\{1\}}). \tag{18}$$

Notice that $H_{k+1}^{\hat{\rho}}$ and $K_{1,k}^{\hat{\rho}}$ should satisfy (17) and (18) to decouple the effects of unknown inputs and measurement noises on the known part of the dynamics. Observing (17) and (18), $H_{k+1}^{\hat{\rho}}$ and $K_{1,k}^{\hat{\rho}}$ may not be affine functions with respect to $\hat{\rho}_k$. In this case, we cannot make the polytopic decomposition of $H_{k+1}^{\hat{\rho}}$ and $K_{1,k}^{\hat{\rho}}$ with respect to $\hat{\rho}_k$ at all.

However, in order to establish stability conditions, we need to obtain the polytopic decomposition of $H_{k+1}^{\hat{\rho}}$ and $K_{1,k}^{\hat{\rho}}$. By considering the bounds of $\hat{\rho}_k$ in (6), we can obtain polytopic sets of $H_{k+1}^{\hat{\rho}}$ and $K_{1,k}^{\hat{\rho}}$ using (17) and (18), respectively. Thus, we assume two new scheduling vectors as functions of $\hat{\rho}_k$:

$$\hat{\rho}_k^h = f_h(\hat{\rho}_k), \tag{19a}$$
$$\hat{\rho}_k^{k1} = f_{k_1}(\hat{\rho}_k) \tag{19b}$$

and can then transform (17) and (18) into their equivalent polytopic decomposition as

$$H_{k+1}^{\hat{\rho}} = \sum_{i=1}^{O_1} \lambda_i(\hat{\rho}_{k+1}^h)\bar{H}_i, \tag{20a}$$
$$K_{1,k}^{\hat{\rho}} = \sum_{i=1}^{O_2} \lambda_i(\hat{\rho}_k^{k1})\bar{K}_{1,i}, \tag{20b}$$

where $\bar{H}_i$ and $\bar{K}_{1,i}$ are the i-th vertex matrices of the polytopic sets of $H_{k+1}^{\hat{\rho}}$ and $K_{1,k}^{\hat{\rho}}$, respectively, and O_1 and O_2 denote the numbers of vertex matrices of the polytopic sets of $H_{k+1}^{\hat{\rho}}$ and $K_{1,k}^{\hat{\rho}}$ subject to (17) and (18), respectively.

Remark 3. *The arbitrarily given $\mathring{H}$ and Y_1 are implicitly included into (20), which can be used as two degrees of freedom to stabilize the corresponding dynamics.*

Thus, based on (17) and (18), (10) can be transformed into

$$\begin{aligned} e_{k+1} =& \Lambda_0 e_k + \Lambda_1 z_k + \Lambda_2 y_k + \Lambda_3 u_k + \tilde{\Lambda}_4 \omega_k \\ &+ \Lambda_5 \eta_{k+1} + \tilde{\Lambda}_6 \eta_k. \end{aligned} \tag{21}$$

By integrating (5) with (21), we can obtain the dynamics

$$\zeta_{k+1} = \Phi_0' \zeta_k + \Phi_1' \mu_k + \Phi_2' \omega_k + \Phi_3' \eta_{k+1} + \Phi_4' \eta_k, \tag{22a}$$
$$\xi_k = \Psi_0 \zeta_k + \Psi_1 \mu_k, \tag{22b}$$
$$y_k = C_k^{\rho} \xi_k + F_k^{\rho} \eta_k, \tag{22c}$$

where ξ_k denotes the output vector[1] of the dynamics with $\xi_k = x_k$ and the other variables and matrices are defined as

$$\Psi_0 = \begin{bmatrix} I & I \end{bmatrix}, \Psi_1 = \begin{bmatrix} H_k^{\hat{\rho}} & \mathbf{0} \end{bmatrix}, \Phi_0' = \begin{bmatrix} N_k^{\hat{\rho}} & \mathbf{0} \\ \Lambda_1 & \Lambda_0 \end{bmatrix}, \Phi_1' = \begin{bmatrix} K_k^{\hat{\rho}} & T_k^{\hat{\rho}} \\ \Lambda_2 & \Lambda_3 \end{bmatrix},$$

$$\Phi_2' = \begin{bmatrix} \mathbf{0} \\ \tilde{\Lambda}_4 \end{bmatrix}, \Phi_3' = \begin{bmatrix} \mathbf{0} \\ \Lambda_5 \end{bmatrix}, \Phi_4' = \begin{bmatrix} \mathbf{0} \\ \tilde{\Lambda}_6 \end{bmatrix}, \zeta_k = \begin{bmatrix} z_k \\ e_k \end{bmatrix}, \mu_k = \begin{bmatrix} y_k \\ u_k \end{bmatrix}.$$

Based on (22), the nominal dynamics of (22) are defined as

$$\bar{\zeta}_{k+1} = \Phi_0' \bar{\zeta}_k + \Phi_1' \mu_k, \tag{23a}$$
$$\bar{\xi}_k = \Psi_0 \bar{\zeta}_k + \Psi_1 \mu_k, \tag{23b}$$
$$\bar{y}_k = C_k^{\rho} \bar{\xi}_k. \tag{23c}$$

By using (22a) and (23a), the dynamics of $\tilde{\zeta}_k = \zeta_k - \bar{\zeta}_k$ are

$$\tilde{\zeta}_{k+1} = \Phi_0' \tilde{\zeta}_k + \Phi_2' \omega_k + \Phi_3' \eta_{k+1} + \Phi_4' \eta_k. \tag{24}$$

Based on the analysis above, robust stability results using the Lyapunov stability theory are obtained in Theorem 1.

Theorem 1. *For the system (1) and parametric matrices of (5), the dynamics (24) is robustly stable if there exist symmetric positive definite matrices P_{i_0} and P_{j_1} such that*

$$\Omega_{i_0,i_1,i_2,j_1,j_2,g_1,g_2,h_1,h_2,l_1,l_2,t_1,t_2,b_1,b_2,d_1,d_2} =$$
$$\begin{bmatrix} {\Phi'}_{0,j_1,g_1,h_1,l_1,t_1}^T P_{i_0} {\Phi'}_{0,j_2,g_2,h_2,l_2,t_2} - P_{j_1} & {\Phi'}_{0,j_1,g_1,h_1,l_1,t_1}^T P_{i_0} \Phi'_{2,i_2,j_2,l_2,b_2,d_2} & {\Phi'}_{0,j_1,g_1,h_1,l_1,t_1}^T P_{i_0} \Phi'_{3,h_2,l_2} & {\Phi'}_{0,j_1,g_1,h_1,l_1,t_1}^T P_{i_0} \Phi'_{4,t_2,d_2} \\ * & {\Phi'}_{2,i_1,j_1,l_1,b_1,d_1}^T P_{i_0} \Phi'_{2,i_2,j_2,l_2,b_2,d_2} & {\Phi'}_{2,i_1,j_1,l_1,b_1,d_1}^T P_{i_0} \Phi'_{3,h_2,l_2} & {\Phi'}_{2,i_1,j_1,l_1,b_1,d_1}^T P_{i_0} \Phi'_{4,t_2,d_2} \\ * & * & {\Phi'}_{3,h_1,l_1}^T P_{i_0} \Phi'_{3,h_2,l_2} & {\Phi'}_{3,h_1,l_1}^T P_{i_0} \Phi'_{4,t_2,d_2} \\ * & * & * & {\Phi'}_{4,t_1,d_1}^T P_{i_0} \Phi'_{4,t_2,d_2} \end{bmatrix} \prec 0 \tag{25}$$

with

$${\Phi'}_{0,j,g,h,l,t} = \begin{bmatrix} N_j & \mathbf{0} \\ A_g - \bar{H}_l C_h A_g - \bar{K}_{1,t} C_g - N_j & A_g - \bar{H}_l C_h A_g - \bar{K}_{1,t} C_g \end{bmatrix},$$

$${\Phi'}_{2,i,j,l,b,d} = \begin{bmatrix} \mathbf{0} \\ \bar{H}_l \mathcal{C}(\mathring{\mathcal{P}}_b \otimes I) E_j - \bar{H}_l \mathcal{C}(\mathring{\mathcal{P}}_b \otimes I)\mathcal{E}(\mathring{\mathcal{P}}_d \otimes I) - \mathcal{E}(\mathring{\mathcal{P}}_d \otimes I) + \\ \bar{H}_l C_i \mathcal{E}(\mathring{\mathcal{P}}_d \otimes I) \end{bmatrix},$$

$${\Phi'}_{3,h,l} = \begin{bmatrix} \mathbf{0} \\ -\bar{H}_l F_h \end{bmatrix}, \ {\Phi'}_{4,t,d} = \begin{bmatrix} \mathbf{0} \\ \bar{K}_{1,t} \mathcal{F}(\mathring{\mathcal{P}}_d \otimes I) \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} C^1 & C^2 & \cdots & C^m \end{bmatrix}, \ \mathcal{E} = \begin{bmatrix} E^1 & E^2 & \cdots & E^m \end{bmatrix},$$

$$\mathcal{F} = \begin{bmatrix} F^1 & F^2 & \cdots & F^m \end{bmatrix},$$

for $\forall\, i,\ j,\ g,\ h,\ b,\ d = 1,\ 2, \cdots,\ N,\ l = 1,\ 2, \cdots,\ O_1$ and $t = 1,\ 2, \cdots,\ O_2$ with the indices defined as

$$N_k^{\hat{\rho}} = \sum_{j=1}^{N} \lambda_{j,k}^{\hat{\rho}} N_j, \ A_k^{\rho} = \sum_{g=1}^{N} \lambda_{g,k}^{\rho} A_g,$$

$$C_{k+1}^{\rho} = \sum_{h=1}^{N} \lambda_{h,k+1}^{\rho} C_h, \ C_k^{\rho} = \sum_{g=1}^{N} \lambda_{g,k}^{\rho} C_g,$$

$$H_{k+1}^{\hat{\rho}} = \sum_{l=1}^{O_1} \lambda_{l,k+1}^{\hat{\rho}^h} \bar{H}_l, \ K_{1,k}^{\hat{\rho}} = \sum_{t=1}^{O_2} \lambda_{t,k}^{\hat{\rho}^{k_1}} \bar{K}_{1,t},$$

$$\sum_{i=1}^{m} C^i \tilde{\rho}_{k+1}^i = \mathcal{C}(\tilde{\rho}_{k+1} \otimes I) = \sum_{b=1}^{N} \lambda_{b,k+1}^{\tilde{\rho}} \mathcal{C}(\mathring{\mathcal{P}}_b \otimes I),$$

[1]Although ξ_k equals to the state x_k and cannot be obtained, it is the formatted output vector of the equivalent augmented dynamics, which will be used for the design of SUIO presented in the following Section IV.

$$\sum_{i=1}^{m} E^i \tilde{\rho}_k^i = \mathcal{E}(\tilde{\rho}_k \otimes I) = \sum_{d=1}^{N} \lambda_{d,k}^{\tilde{\rho}} \mathcal{E}(\mathring{\mathcal{P}}_d \otimes I),$$
$$\sum_{i=1}^{m} F^i \tilde{\rho}_k^i = \mathcal{F}(\tilde{\rho}_k \otimes I) = \sum_{d=1}^{N} \lambda_{d,k}^{\tilde{\rho}} \mathcal{F}(\mathring{\mathcal{P}}_d \otimes I),$$

where the indices i, j, g, h, l, t, b and d take i_0 (i_1 or i_2), j_1 (or j_2), g_1 (or g_2), h_1 (or h_2), l_1 (or l_2), t_1 (or t_2), b_1 (or b_2) and d_1 (or d_2), respectively, $\sum_{b=1}^{N} \lambda_{b,k+1}^{\tilde{\rho}} = 1$ with $0 \leq \lambda_{b,k+1}^{\tilde{\rho}} \leq 1$ (similar to $\lambda_{d,k}^{\tilde{\rho}}$), and $\mathring{\mathcal{P}}_b$ is the b-th vertex of the bounding set $\tilde{\mathcal{P}}$ (similar to $\mathring{\mathcal{P}}_d$).

Proof. See Proof of Theorem 1 in Appendix. □

Although Theorem 1 gives robust stability conditions with respect to both parametric uncertainties, unknown inpus and noises, it is observed that these conditions are quite conservative even provide no solutions to the parametric matrices of (5) in some cases. Thus, considering that the robust stability conditions in Theorem 1 are only sufficient conditions and that the uncertain factors η_k and η_{k+1} in (24) are bounded, we further make a stability analysis for the nominal system:

$$\tilde{\zeta}_{k+1} = \Phi_0' \tilde{\zeta}_k, \tag{28}$$

which omits the effect of the unknown inputs and noises. Here, stability results in Lemmas 2 and 3 taken from [11] and [30] are first recalled for stability analysis of (28).

Lemma 2. *Considering the discrete LPV system:*

$$x_{k+1} = \begin{bmatrix} \mathcal{A}_{11,k}^{\hat{\rho}} & \mathbf{0} \\ \mathcal{A}_{21,k}^{\hat{\rho}} & \mathcal{A}_{22,k}^{\hat{\rho}} \end{bmatrix} x_k \tag{29}$$

and assume that $\mathcal{A}_{11,k}^{\hat{\rho}}$ and $\mathcal{A}_{22,k}^{\hat{\rho}}$ are quadratically stable[2]*, respectively, then the LPV system (29) is quadratically stable.*

Lemma 3. *The dynamics $x_{k+1} = \mathcal{A}_k^{\hat{\rho}} x_k$ is stable if and only if there exists symmetric definite matrices S_i, S_j and matrices G_i such that*

$$\begin{bmatrix} S_j & * \\ G_i \mathcal{A}_i & G_i + G_i^T - S_i \end{bmatrix} \succ 0, \ \forall\, i,\ j = 1, 2, \ , N, \tag{30}$$

where the symbol $*$ denotes the transpose of $G_i \mathcal{A}_i$. In this case, the time-varying parameter dependent Lyapunov function for the stability is given as $V(x_k, \lambda_{i,k}^{\hat{\rho}}) = x_k^T S_k^{\hat{\rho}} x_k$ with $S_k^{\hat{\rho}} = \sum_{i=1}^{N} \lambda_{i,k}^{\hat{\rho}} S_i$, $\sum_{i=1}^{N} \lambda_{i,k}^{\hat{\rho}} = 1$ and $0 \leq \lambda_{i,k}^{\hat{\rho}} \leq 1$.

It can be observed that Φ_0' is a triangular block matrix. By using the results in Lemmas 2 and 3, stability conclusions for (28) are obtained in Theorem 2. It is clear that Theorem 2 is less conservative than Theorem 1. Thus, we treat Theorem 2 as more pragmatic stability conditions than Theorem 1, which may be able to reduce the conservatism when designing the parametric matrices of (5) for robust FD.

Theorem 2. *For the system (1) and a group of parametric matrices of (5), the dynamics (28) is stable if there exist symmetric definite matrices $S_{1,i}$, $S_{1,j}$, $S_{2,i}$, $S_{2,j}$, and matrices $G_{1,i}$ and $G_{2,i}$ such that*

$$\begin{bmatrix} S_{1,j} & * \\ G_{1,i} N_i & G_{1,i} + G_{1,i}^T - S_{1,i} \end{bmatrix} \succ 0, \tag{31a}$$

$$\begin{bmatrix} S_{2,j} & * \\ G_{2,i}(A_i - \bar{H}_g C_h A_i - \bar{K}_{1,l} C_i) & G_{2,i} + G_{2,i}^T - S_{2,i} \end{bmatrix} \succ 0 \tag{31b}$$

$\forall i, j, h = 1, 2, \cdots, N$, $g = 1, 2, \cdots, O_1$ and $l = 1, 2, \cdots, O_2$, with the indices defined as $N_k^{\hat{\rho}} = \sum_{i=1}^{N} \lambda_{i,k}^{\hat{\rho}} N_i$, $A_k^{\rho} = \sum_{i=1}^{N} \lambda_{i,k}^{\rho} A_i$, $C_k^{\rho} = \sum_{i=1}^{N} \lambda_{i,k}^{\rho} C_i$, $C_{k+1}^{\rho} = \sum_{h=1}^{N} \lambda_{h,k+1}^{\rho} C_h$, $H_{k+1}^{\hat{\rho}} = \sum_{g=1}^{O_1} \lambda_{g,k+1}^{\hat{\rho}^h} \bar{H}_g$ and $K_{1,k}^{\hat{\rho}} = \sum_{l=1}^{O_2} \lambda_{l,k}^{\hat{\rho}^{k_1}} \bar{K}_{1,l}$, where the symbol $*$ denotes the transpose of $G_{1,i} N_i$ in (31a) and $G_{2,i}(A_i - \bar{H}_g C_h A_i - \bar{K}_{1,l} C_i)$ in (31b), respectively.

Proof. See Proof of Theorem 2 in Appendix. □

C. The Second Situation

In Situation 2, the basic idea is similar to Situation 1. For some LPV systems (1), the effect of the term η_{k+1} on the FD conservatism may be higher than that of the term ω_k, in this case, choosing to actively decouple η_{k+1} (i.e., corresponding to (13f)) instead of ω_k (i.e., corresponding to (13e)) is possible to provide better robust FD performance, which is the main motivation of this subsection. Thus, in Situation 2, we design $H_k^{\hat{\rho}}$ and $K_{1,k}^{\hat{\rho}}$ such that $\hat{\Lambda}_5 = \mathbf{0}$ and $\hat{\Lambda}_6 = \mathbf{0}$ (see (13f) and (13g)). By using Lemma 1, the solution of $\hat{\Lambda}_5 = \mathbf{0}$ is

$$H_{k+1}^{\hat{\rho}} = Y_2(I - F_{k+1}^{\hat{\rho}}(F_{k+1}^{\hat{\rho}})^{\{1\}}). \tag{32}$$

Similarly, by substituting (18) and (32) into (10), the dynamics (10) can be transformed into

$$\begin{aligned} e_{k+1} = &\Lambda_0 e_k + \Lambda_1 z_k + \Lambda_2 y_k + \Lambda_3 u_k + \Lambda_4 \omega_k \\ &+ \tilde{\Lambda}_5 \eta_{k+1} + \tilde{\Lambda}_6 \eta_k. \end{aligned} \tag{33}$$

The integrated dynamics of (5) and (33) can be obtained in the similar way. However, in Situation 2, the matrices Φ_2' and Φ_3' in (22) should be modified into

$$\Phi_2'' = \begin{bmatrix} \mathbf{0} \\ \Lambda_4 \end{bmatrix}, \ \Phi_3'' = \begin{bmatrix} \mathbf{0} \\ \tilde{\Lambda}_5 \end{bmatrix}. \tag{34}$$

Furthermore, we can design the dynamics similar with (23) corresponding to Situation 2 and the error between the integrated dynamics and the nominal dynamics of this integrated dynamics can be derived as the same mathematical form with (24). However, we should consider the new expressions of Φ_2'' and Φ_3'' in the dynamics of the error as given in (34).

Moreover, a stability theorem can also be established for Situation 2 as done in Section III-B by considering both (18) and (32) to design parametric matrices for stability guarantees.

Remark 4. *The constraints $\hat{\Lambda}_4 = \mathbf{0}$ and $\hat{\Lambda}_6 = \mathbf{0}$ for partially decoupling the effects of unknown inputs and measurement noises on the known part (13) of (10) have already been integrated into the stability conditions in Theorems 1 and 2 by substituting (20) (i.e., (17) and (18)) into (10) and consequently obtaining (21), (22a), (24) and (28), which provides more design degrees of freedom to the parametric matrices of (5). Please also see [4], [5] for more details on the point integrating unknown inputs decoupling conditions with stability conditions in the design of UIOs.*

[2]The expression on the stability of a matrix $\mathcal{A}_k^{\hat{\rho}}$ means the stability of the system $x_{k+1} = \mathcal{A}_k^{\hat{\rho}} x_k$.

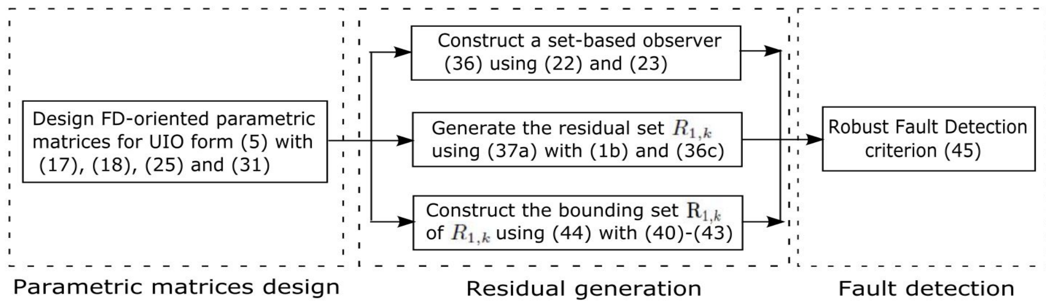

Fig. 1. Flow chart of robust FD

D. Robust Fault Detection

Due to the fact that Situations 1 and 2 are similar, this subsection only takes Situation 1 as an example to propose a novel robust FD method for the general LPV system (1).

In Section III-B, since Φ_0' and Φ_1' are unknown, ξ_k and ζ_k are actually unknown signals. This implies that the dynamics (23) cannot be used as an observer to generate available residual signals. However, we can still define an intermediate signal r_k by using (22) and (23) for residual analysis:

$$r_k = y_k - \bar{y}_k = C_k^{\rho}\Psi_0\tilde{\zeta}_k + F_k^{\rho}\eta_k, \tag{35}$$

where $\tilde{\zeta}_k = \zeta_k - \bar{\zeta}_k$ is subject to (24). Because r_k is unknown as well, we define a set-based observer using (23) as

$$\begin{aligned}\hat{\Gamma}_{k+1}' &= \mathbf{\Phi}_0'\hat{\Gamma}_k' \oplus \{\mathbf{\Phi}_1'\mu_k\}, &(36a)\\ \hat{\Xi}_k' &= \Psi_0\hat{\Gamma}_k' \oplus \{\Psi_1\mu_k\}, &(36b)\\ \hat{Y}_{1,k} &= C_k^{\hat{\rho}}\hat{\Xi}_k', &(36c)\\ \hat{Y}_{2,k} &= C_k^{\rho}\hat{\Xi}_k', &(36d)\end{aligned}$$

where $\mathbf{\Phi}_0'$ and $\mathbf{\Phi}_1'$ are the interval matrices of Φ_0' and Φ_1', respectively (note that $\mathbf{\Phi}_0'$ and $\mathbf{\Phi}_1'$ can be computed in the same way with $\mathbf{\Phi}_0$ and $\mathbf{\Phi}_1$ derived in (58) in the following Section IV by considering the bounds $\tilde{\mathcal{P}}^i$ of $\tilde{\rho}_k^i$ for all $i = 1, 2, \cdots, m$ for $\tilde{\Lambda}_0$, $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$ and $\tilde{\Lambda}_3$ in (14)), $\hat{\Gamma}_k'$ and $\hat{\Xi}_k'$ are the estimated sets of $\bar{\zeta}_k$ and $\bar{\xi}_k$, respectively, and $\hat{Y}_{1,k}$ and $\hat{Y}_{2,k}$ are available and unavailable sets, respectively, both defined for the implementation of the proposed robust FD method.

Based on (22) and (36), the two following residual sets are defined for the implementation of robust FD:

$$\begin{aligned}R_{1,k} &= \{y_k\} \oplus (-\hat{Y}_{1,k}), &(37a)\\ R_{2,k} &= \{y_k\} \oplus (-\hat{Y}_{2,k}), &(37b)\end{aligned}$$

where $R_{1,k}$ is known but $R_{2,k}$ is unknown. Moreover, using the results in (8) and (37), the mathematical relationship between $R_{1,k}$ and $R_{2,k}$ can be derived as

$$\begin{aligned}R_{1,k} &= \{y_k\} \oplus (-\hat{Y}_{1,k}) = \{y_k\} \oplus (-C_k^{\hat{\rho}}\hat{\Xi}_k')\\ &= \{y_k\} \oplus (-C_k^{\rho} - \sum_{i=1}^{m} C^i\tilde{\rho}_k^i)\hat{\Xi}_k'\\ &= R_{2,k} \oplus (-\sum_{i=1}^{m} C^i\tilde{\rho}_k^i)\hat{\Xi}_k'.\end{aligned} \tag{38}$$

As seen in (38), it is necessary to further analyze the feature of $R_{2,k}$ to understand the feature of $R_{1,k}$ in detail. Thus, according to (8), (22), (36) and (37b), we can have

$$\begin{aligned}R_{2,k} &= \{y_k\} \oplus (-\hat{Y}_{2,k}) = C_k^{\rho}\{\{x_k\} \oplus (-\hat{\Xi}_k')\} \oplus \{F_k^{\rho}\eta_k\}\\ &= C_k^{\rho}\{\{\xi_k\} \oplus (-\hat{\Xi}_k')\} \oplus \{F_k^{\rho}\eta_k\} = C_k^{\rho}\Psi_0\Gamma_k' \oplus \{F_k^{\rho}\eta_k\}\\ &= (C_k^{\hat{\rho}} - \sum_{i=1}^{m} C^i\tilde{\rho}_k^i)\Psi_0\Gamma_k' \oplus \{(F_k^{\hat{\rho}} - \sum_{i=1}^{m} F^i\tilde{\rho}_k^i)\eta_k\}\end{aligned} \tag{39}$$

with

$$\Gamma_k' = \{\zeta_k\} \oplus (-\hat{\Gamma}_k'). \tag{40}$$

Furthermore, considering the bounds $\mathbf{\Phi}_0'$, $\mathbf{\Phi}_1'$, $\mathbf{\Phi}_2'$, $\mathbf{\Phi}_3'$, $\mathbf{\Phi}_4'$, W, V of Φ_0', Φ_1', Φ_2', Φ_3', Φ_4', ω_k and η_k (or η_{k+1}), the set-based dynamics of (22a) can be derived as

$$\check{\Gamma}_{k+1} = \mathbf{\Phi}_0'\check{\Gamma}_k \oplus \mathbf{\Phi}_1'\mu_k \oplus \mathbf{\Phi}_2'W \oplus \mathbf{\Phi}_3'V \oplus \mathbf{\Phi}_4'V, \tag{41}$$

where if $\zeta_0 \in \check{\Gamma}_0$ is given, we can have $\zeta_k \in \check{\Gamma}_k$ for all $k > 0$. Note that the computation of $\mathbf{\Phi}_2'$, $\mathbf{\Phi}_3'$, $\mathbf{\Phi}_4'$ is similar with the computation of $\mathbf{\Phi}_0'$ and $\mathbf{\Phi}_1'$ introduced above. Additionally, considering that $\mathbf{\Phi}_0'$, $\mathbf{\Phi}_1'$, $\mathbf{\Phi}_2'$, $\mathbf{\Phi}_3'$ and $\mathbf{\Phi}_4'$ are all interval matrices, the methods given in Properties 3 and 4 in Appendix are used to implement the zonotopic propagation in (41).

Thus, according to (36a), (40) and (41), we could construct a bounding set $\mathbf{\Gamma}_k$ for Γ_k', i.e.,

$$\Gamma_k' \subseteq \mathbf{\Gamma}_k = \check{\Gamma}_k \oplus (-\hat{\Gamma}_k'). \tag{42}$$

Moreover, by considering $\mathbf{\Gamma}_k$, $\tilde{\mathcal{P}}^i$ and V to replace Γ_k', $\tilde{\rho}_k^i$ and η_k in (39), we can obtain the bounding set of $R_{2,k}$ as

$$\begin{aligned}R_{2,k} \subset \mathbf{R}_{2,k} &= (C_k^{\hat{\rho}} - \sum_{i=1}^{m} C^i\tilde{P}^i)\Psi_0\mathbf{\Gamma}_k\\ &\quad \oplus (F_k^{\hat{\rho}} - \sum_{i=1}^{m} F^i\tilde{P}^i)V.\end{aligned} \tag{43}$$

According to (38), since $R_{2,k}$ is an unknown set, a bounding set $\mathbf{R}_{1,k}$ of $R_{1,k}$ can be further constructed by using the bounding set $\mathbf{R}_{2,k}$ of $R_{2,k}$ as

$$\mathbf{R}_{1,k} = \mathbf{R}_{2,k} \oplus (-\sum_{i=1}^{m} C^i\tilde{P}^i)\hat{\Xi}_k', \tag{44}$$

i.e.,

$$R_{1,k} \subseteq \mathbf{R}_{1,k}. \tag{45}$$

Note that since both $R_{1,k}$ and $\mathbf{R}_{1,k}$ can be computed online by using (37a), (43) and (44), respectively, the inclusion (45) can be used as a robust FD criterion of the LPV system (1). This implies that, as long as (45) is violated at a time instant, it is sure that the LPV system (1) has become faulty. Otherwise, it is considered that the system is still healthy.

Moreover, since the FD criterion (45) is to test the inclusion between two real-time available sets, the unknown measurement errors of scheduling variables can be effectively handled, which is different from the traditional robust FD methods. In order to help the readers understand the logic of the proposed FD, a flow chart of robust FD is shown in Figure 1.

Remark 5. *Since this paper only considers robust FD while fault isolation is not included, all types of faults can be detected as long as the FD criterion (45) is violated. Thus, without loss of generality, only actuator faults will be used to show the effectiveness of the proposed robust FD method in the illustrative example given in Section V.*

IV. Design of SUIO for LPV Systems

Different from Section III, this section designs an SUIO for a reduced version of the LPV systems (1) under the unknown input decoupling condition given in Assumption 2.

A. Unknown Input Decoupling Condition for LPV Systems

In Section II-C, the dynamics of the SE error of the observer (5) for the LPV system (1) is obtained in (10). It can be observed that the parametric matrix Λ_4 derived in (11e) should be designed to be zero such that the unknown input vector ω_k is actively decoupled by (5).

Assumption 2. *For the LPV system (1), C_k^{ρ} is a known constant matrix denoted as C and E_k^{ρ} can be factorized into $E_k^{\rho} = E\bar{E}_k^{\rho}$ (or E_k^{ρ} is also a known constant matrix denoted as E) with rank$(E) =$ rank(CE), where E is a known constant matrix and $\bar{E}_k^{\rho}$ is a scheduling vector-dependent matrix.*

To explain the proposed SUIO design method, a general SUIO definition referring to [36] is stated in Definition 1.

Definition 1. *For a system under effects of uncertain factors (i.e., unknown inputs, process disturbances, measurement/modeling errors and measurement noises), an SUIO is defined as a robust SE observer, which is able to completely/partially remove the effects of unknown inputs on robust SE, simultaneously use the set theory to bound the effects of other uncertain factors and eventually obtain robust SE sets always including the real system states.*

The condition in Assumption 2 is the unknown input decoupling condition of the LPV system (1) and is only made for the design of SUIO in Section IV. Under this assumption, the LPV system (1) is reduced into

$$x_{k+1} = A_k^{\rho} x_k + B_k^{\rho} u_k + E\bar{E}_k^{\rho}\omega_k, \tag{46a}$$
$$y_k = Cx_k + F_k^{\rho}\eta_k. \tag{46b}$$

This implies that, under Assumption 2 and Definition 1, we could design an SUIO for the LPV system (46) and the mathematical form of the UIO (5) should also be reduced into

$$z_{k+1} = N(\hat{\rho}_k)z_k + T(\hat{\rho}_k)u_k + K(\hat{\rho}_k)y_k, \tag{47a}$$
$$\hat{x}_k = z_k + Hy_k, \tag{47b}$$

where the parametric matrix H is a constant matrix.

Correspondingly, the dynamics of the SE error of (47) is further transformed into

$$\begin{aligned} e_{k+1} =& \Lambda_0' e_k + \Lambda_1' z_k + \Lambda_2' y_k + \Lambda_3' u_k + \Lambda_4' \omega_k \\ &+ \Lambda_5' \eta_{k+1} + \Lambda_6' \eta_k, \end{aligned} \tag{48}$$

where

$$\begin{aligned} \Lambda_0' =& \hat{\Lambda}_0' + \tilde{\Lambda}_0',\ \Lambda_1' = \hat{\Lambda}_1' + \tilde{\Lambda}_1',\ \Lambda_2' = \hat{\Lambda}_2' + \tilde{\Lambda}_2',\ \Lambda_3' = \hat{\Lambda}_3' + \tilde{\Lambda}_3', \\ \Lambda_4' =& \hat{\Lambda}_4' + \tilde{\Lambda}_4',\ \Lambda_5' = \hat{\Lambda}_5' + \tilde{\Lambda}_5',\ \Lambda_6' = \hat{\Lambda}_6' + \tilde{\Lambda}_6'. \end{aligned} \tag{49}$$

The parametric matrices of the dynamics in (48) and (49) are obtained by reducing (11) and (14) into

$$\hat{\Lambda}_0' = A_k^{\hat{\rho}} - HCA_k^{\hat{\rho}} - K_{1,k}^{\hat{\rho}}C, \tag{50a}$$
$$\hat{\Lambda}_1' = A_k^{\hat{\rho}} - HCA_k^{\hat{\rho}} - K_{1,k}^{\hat{\rho}}C - N_k^{\hat{\rho}}, \tag{50b}$$
$$\hat{\Lambda}_2' = (A_k^{\hat{\rho}} - HCA_k^{\hat{\rho}} - K_{1,k}^{\hat{\rho}}C)H - K_{2,k}^{\hat{\rho}}, \tag{50c}$$
$$\hat{\Lambda}_3' = B_k^{\hat{\rho}} - T_k^{\hat{\rho}} - HCB_k^{\hat{\rho}}, \tag{50d}$$
$$\hat{\Lambda}_4' = E - HCE, \tag{50e}$$
$$\hat{\Lambda}_5' = -HF_{k+1}^{\hat{\rho}}, \tag{50f}$$
$$\hat{\Lambda}_6' = -K_{1,k}^{\hat{\rho}}F_k^{\hat{\rho}} \tag{50g}$$

and

$$\tilde{\Lambda}_0' = -\sum_{i=1}^{m}(A^i - HCA^i)\tilde{\rho}_k^i, \tag{51a}$$
$$\tilde{\Lambda}_1' = -\sum_{i=1}^{m}(A^i - HCA^i)\tilde{\rho}_k^i, \tag{51b}$$
$$\tilde{\Lambda}_2' = -\sum_{i=1}^{m}(A^i - HCA^i)H\tilde{\rho}_k^i, \tag{51c}$$
$$\tilde{\Lambda}_3' = -\sum_{i=1}^{m}(B^i - HCB^i)\tilde{\rho}_k^i, \tag{51d}$$
$$\tilde{\Lambda}_4' = \mathbf{0}, \tag{51e}$$
$$\tilde{\Lambda}_5' = \sum_{i=1}^{m} HF^i\tilde{\rho}_k^i, \tag{51f}$$
$$\tilde{\Lambda}_6' = \sum_{i=1}^{m} K_{1,k}^{\hat{\rho}}F^i\tilde{\rho}_k^i. \tag{51g}$$

For further analysis, an augmented dynamics integrating (47a) with (48) can be obtained as

$$\zeta_{k+1} = \Phi_0\zeta_k + \Phi_1\mu_k + \Phi_2\omega_k + \Phi_3\eta_{k+1} + \Phi_4\eta_k, \tag{52a}$$
$$\xi_k = \Psi_0\zeta_k + \Psi_1\mu_k, \tag{52b}$$

with

$$\Phi_0 = \begin{bmatrix} N_k^{\hat{\rho}} & \mathbf{0} \\ \Lambda_1' & \Lambda_0' \end{bmatrix}, \Phi_1 = \begin{bmatrix} K_k^{\hat{\rho}} & T_k^{\hat{\rho}} \\ \Lambda_2' & \Lambda_3' \end{bmatrix},$$
$$\Phi_2 = \begin{bmatrix} \mathbf{0} \\ \Lambda_4' \end{bmatrix}, \Phi_3 = \begin{bmatrix} \mathbf{0} \\ \Lambda_5' \end{bmatrix}, \Phi_4 = \begin{bmatrix} \mathbf{0} \\ \Lambda_6' \end{bmatrix}.$$

From (50) and (51), it is almost impossible to remove the effect of z_k, y_k and u_k on the dynamics of the SE error. But it is possible to design Λ_4' to be zero such that the effect of unknown inputs contained in ω_k on robust SE is removed. Moreover, based on the results in [10], the condition to solve

$$\Lambda_4' = \mathbf{0} \tag{53}$$

is given as $\mathrm{rank}(E) = \mathrm{rank}(CE)$ as shown in Assumption 2 and the solution of (53) can be computed as

$$H = E[(CE)^T CE]^{-1}(CE)^T$$

$$+ \bar{H}\{I - CE[(CE)^T CE]^{-1}(CE)^T\}, \tag{54}$$

where $\bar{H}$ is an arbitrary compatible matrix.

B. SUIO for Robust State Estimation

Due to the measurement error $\tilde{\rho}_k$, some parametric matrices of the augmented dynamics (52) are unknown. Thus, it is difficult to directly design an observer for (52) to obtain robust SE. In order to solve this problem, this paper proposes to use the nominal system of (52) to divide the SE problem of the system (46) into two parts: the output estimation of this nominal system and the difference estimation between the system (52) and its nominal system. As a result, robust SE of the system (46) can be obtained by using the estimations of these two parts. The nominal system is defined as

$$\bar{\zeta}_{k+1} = \Phi_0 \bar{\zeta}_k + \Phi_1 \mu_k, \tag{55a}$$
$$\bar{\xi}_k = \Psi_0 \bar{\zeta}_k + \Psi_1 \mu_k, \tag{55b}$$

where $\bar{\zeta}_k$ and $\bar{\xi}_k$ are the state and output vectors of (55). Thus, under the condition (53) and Assumption 2, by using (52a) and (55a), the dynamics of $\tilde{\zeta}_k = \zeta_k - \bar{\zeta}_k$ can be derived as

$$\tilde{\zeta}_{k+1} = \Phi_0 \tilde{\zeta}_k + \Phi_3 \eta_{k+1} + \Phi_4 \eta_k, \tag{56}$$

where Φ_0 and Φ_1 can be rewritten into

$$\Phi_0 = \begin{bmatrix} N_k^{\hat{\rho}} & \mathbf{0} \\ \hat{\Lambda}_1' & \hat{\Lambda}_0' \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} 0 & 0 \\ HCA^i - A^i & HCA^i - A^i \end{bmatrix} \tilde{\rho}_k^i, \tag{57a}$$
$$\Phi_1 = \begin{bmatrix} K_k^{\hat{\rho}} & T_k^{\hat{\rho}} \\ \hat{\Lambda}_2' & \hat{\Lambda}_3' \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} 0 & 0 \\ (HCA^i - A^i)H & HCB^i - B^i \end{bmatrix} \tilde{\rho}_k^i. \tag{57b}$$

Remark 6. *The equivalent transformations of Φ_0 and Φ_1 given in (57) are mainly used to reduce the computational conservatism of interval operations (more details on interval operations can be found in [23]).*

In (51), since $\tilde{\rho}_k^i$ is unknown, Φ_0 and Φ_1 are also unknown. Thus, we turn to the set theory by considering the bounds of $\tilde{\rho}_k^i$ given in (4) to compute the bounds of Φ_0 and Φ_1 as

$$\mathbf{\Phi}_0 = \begin{bmatrix} N_k^{\hat{\rho}} & \mathbf{0} \\ \hat{\Lambda}_1' & \hat{\Lambda}_0' \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} 0 & 0 \\ HCA^i - A^i & HCA^i - A^i \end{bmatrix} \tilde{\mathcal{P}}_i, \tag{58a}$$
$$\mathbf{\Phi}_1 = \begin{bmatrix} K_k^{\hat{\rho}} & T_k^{\hat{\rho}} \\ \hat{\Lambda}_2' & \hat{\Lambda}_3' \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} 0 & 0 \\ (HCA^i - A^i)H & HCB^i - B^i \end{bmatrix} \tilde{\mathcal{P}}_i, \tag{58b}$$

respectively, where $\mathbf{\Phi}_0$ and $\mathbf{\Phi}_1$ are two interval matrices.

Based on (55) and (58), we can design a set-based observer for the nominal system (55):

$$\hat{\Gamma}_{k+1} = \mathbf{\Phi}_0 \hat{\Gamma}_k \oplus \mathbf{\Phi}_1 \mu_k, \tag{59a}$$
$$\hat{\Xi}_k = \Psi_0 \hat{\Gamma}_k \oplus \Psi_1 \mu_k, \tag{59b}$$

where $\hat{\Gamma}_k$ and $\hat{\Xi}_k$ are the estimated sets of $\bar{\zeta}_k$ and $\bar{\xi}_k$, respectively, (i.e., $\bar{\zeta}_k \in \hat{\Gamma}_k$ and $\bar{\xi}_k \in \hat{\Xi}_k$).

For a balance between computational complexity and accuracy, zonotopes are used for set computations in this paper and the definition of zonotopes are given in Definition 2 in Appendix. Note that, by using Properties 1, 2, 3, 4 and 5 in Appendix, the set-based observer (59) can be implemented (especially, the methods in Properties 3 and 4 are used to implement the zonotopic propagation (59) and (60) similar to (41)). But, in order to obtain a zonotopic estimated set, we have to do a zonotopic outer-approximation at each step as shown in Property 5. For simplicity, we still use the same notations $\hat{\Gamma}_k$ and $\hat{\Xi}_k$ to denote their zonotopic outer-approximations for (59) here and in the following part of this paper (see Property 4). Additionally, as k increases, the order of estimated zonotopes also increases. Thus, Property 5 should be used to control the order in the implementation of (59).

Based on the results above and under the decoupling conditions of unknown inputs given in (53) and Assumption 2, a novel SUIO is designed for the LPV system (46) in Theorem 3.

Theorem 3. *For the LPV system (46), the dynamics*

$$\mathbf{\Gamma}_{k+1} = \mathbf{\Phi}_0 \mathbf{\Gamma}_k \oplus \mathbf{\Phi}_3 V \oplus \mathbf{\Phi}_4 V, \tag{60a}$$
$$\hat{X}_k = \hat{\Xi}_k \oplus \Psi_0 \mathbf{\Gamma}_k \tag{60b}$$

is an SUIO of (46), where $\mathbf{\Phi}_3$ and $\mathbf{\Phi}_4$ are interval matrices of Φ_3 and Φ_4 for $\tilde{\rho}_k \in \tilde{P}$, respectively, and $\mathbf{\Gamma}_k$ is the set of $\tilde{\zeta}_k$. Moreover, for initial vectors ζ_0, $\bar{\zeta}_0$ and initial set $\mathbf{\Gamma}_0$, as long as $\tilde{\zeta}_0 \in \mathbf{\Gamma}_0$ is given, it follows

$$x_k \in \hat{X}_k, \tag{61}$$

where $\hat{X}_k$ is the robust SE set at time instant k.

Proof. See Proof of Theorem 3 in Appendix. □

Since the zonotopic outer-approximation errors are introduced into the SUIO (see Property 4), even though the observer (60) is designed to be stable, robust SE is still possible to diverge due to the accumulation of errors (see the wrapping effect of set-based observers in [21] for details).

C. Stability Analysis

The Lyapunov stability theory is used to analyze robust stability of the SUIO and the corresponding robust stability results are obtained in Theorem 4.

Theorem 4. *For the system (46) and a group of parametric matrices of (47), the dynamics (56) is robustly stable if there exist symmetric positive definite matrices P_i and P_{j_1} such that*

$$\Omega_{i,j_1,j_2,g_1,g_2,h_1,h_2} = \begin{bmatrix} \Phi_{0,j_1,g_1}^T P_i \Phi_{0,j_2,g_2} - P_{j_1} & \Phi_{0,j_1,g_1}^T P_i \Phi_{3,h_1} & \Phi_{0,j_1,g_1}^T P_i \Phi_{4,j_2,g_2} \\ * & \Phi_{3,h_1}^T P_i \Phi_{3,h_2} & \Phi_{3,h_1}^T P_i \Phi_{4,j_1,g_1} \\ * & * & \Phi_{4,j_1,g_1}^T P_i \Phi_{4,j_2,g_2} \end{bmatrix} \prec 0 \tag{62}$$

with

$$\Phi_{0,j,g} = \begin{bmatrix} N_j & \mathbf{0} \\ A_g - HCA_g - K_{1,j}C - N_j & A_g - HCA_g - K_{1,j}C \end{bmatrix}, \tag{63a}$$
$$\Phi_{3,h} = \begin{bmatrix} \mathbf{0} \\ -HF_h \end{bmatrix}, \tag{63b}$$
$$\Phi_{4,j,g} = \begin{bmatrix} \mathbf{0} \\ -K_{1,j}F_g \end{bmatrix}, \tag{63c}$$

for $\forall\ i,\ j_1,\ j_2,\ g_1,\ g_2,\ h_1,\ h_2 = 1,\ 2, \cdots,\ N$, with the indices defined as $N_k^{\hat{\rho}} = \sum_{j=1}^{N} \lambda_{j,k}^{\hat{\rho}} N_j$, $A_k^{\rho} = \sum_{g=1}^{N} \lambda_{g,k}^{\rho} A_g$, $K_{1,k}^{\hat{\rho}} = \sum_{j=1}^{N} \lambda_{j,k}^{\hat{\rho}} K_{1,j}$, $F_{k+1}^{\rho} = \sum_{h=1}^{N} \lambda_{h,k+1}^{\rho} F_h$ and $F_k^{\rho} =$

$\sum_{g=1}^{N} \lambda_{g,k}^{\rho} F_g$, where the symbol $\prec$ denotes negative definite, the symbols $*$ denote the transposes of the matrix elements in the corresponding symmetric positions, and the indices j, g and h take j_1 (or j_2), g_1 (or g_2) and h_1 (or h_2), respectively.

Proof. See Proof of Theorem 4 in Appendix. □

The conditions in Theorem 4 can be used to compute the vertex matrices of $N_k^{\hat{\rho}}$ and $K_{1,k}^{\hat{\rho}}$ to guarantee robust stability of the SUIO. But due to the observed conservatism of these conditions, they may provide even no solutions for some applications. Thus, similar with Theorem 2, we also propose to analyze the stability of the nominal dynamics to find some less conservative but pragmatic stability conditions that may be able to guarantee robust stability of the SUIO:

$$\tilde{\zeta}_{k+1} = \Phi_0 \tilde{\zeta}_k. \tag{64}$$

By using the results in Lemmas 2 and 3, a stability conclusion of (64) is presented in Theorem 5.

Theorem 5. *For the system (46) and a group of parametric matrices of (47), the dynamics (64) is stable if there exists symmetric definite matrices $S_{1,i}$, $S_{1,j}$, $S_{2,i}$, $S_{2,j}$, and matrices $G_{1,i}$ and $G_{2,i}$ such that*

$$\begin{bmatrix} S_{1,j} & * \\ G_{1,i}N_i & G_{1,i}+G_{1,i}^T-S_{1,i} \end{bmatrix} \succ 0, \tag{65a}$$

$$\begin{bmatrix} S_{2,j} & * \\ G_{2,i}(A_g-HCA_g-K_{1,i}C) & G_{2,i}+G_{2,i}^T-S_{2,i} \end{bmatrix} \succ 0, \tag{65b}$$

$\forall\ i,\ j,\ g = 1, 2, \cdots, N$, with the indices defined as $N_k^{\hat{\rho}} = \sum_{i=1}^{N} \lambda_{i,k}^{\hat{\rho}} N_i$, $A_k^{\rho} = \sum_{g=1}^{N} \lambda_{g,k}^{\rho} A_g$ and $K_{1,k}^{\hat{\rho}} = \sum_{i=1}^{N} \lambda_{i,k}^{\hat{\rho}} K_{1,i}$, where the symbol $\succ$ denotes positive definite and the symbol $*$ denotes the transpose of $G_{1,i}N_i$ in (65a) and $G_{2,i}(A_g - HCA_g - K_{1,i}C)$ in (65b), respectively.

Proof. See Proof of Theorem 5 in Appendix. □

Remark 7. *The conditions in Theorem 4 are sufficient to guarantee robust stability of the original dynamics, while the conditions in Theorem 5 are sufficient to guarantee the stability of the nominal dynamics. Since the noises are omitted in the nominal dynamics, it is clear that solving the conditions in Theorem 5 is less conservative than that in Theorem 4 and it is more possible for the former to have a feasible solution than the latter. Thus, although the conditions in Theorem 5 cannot guarantee robust stability of the original dynamics, it is still possible for us to use them to obtain a group of parametric matrices for (5) that are able to stabilize the SUIO for robust SE. Thus, the relation of Theorems 4 and 5 are complementary, where the former has more theoretic value, while the latter has more pragmatic value. For a particular application, if solving the conditions in Theorem 4 is computationally infeasible, we could turn to solving the conditions in Theorem 5 as an alternative way to find a group of parametric matrices able to robustly stabilize the SUIO. In Section III, Theorems 1 and 2 also have the same relationship with each other.*

V. Illustrative Examples

Considering the implemental similarity of robust FD in Section III and robust SE in Section IV, this section only uses a vehicle example taken from [13] to illustrate the proposed robust FD method of the LPV system (1) for saving space.

TABLE I
Parameters of Vehicle Model

m, I_z	mass and inertia ($991kg$, $1574kgm^2$)
c_f	front cornering stiffness($41.6kN/rad$)
c_r	rear cornering stiffness($47.13kN/rad$)
l_f	distance from CG to front axle($1.00m$)
l_r	distance from CG to rear axle($1.46m$)
l_ω	distance of wind force action($0.40m$)

In the vehicle example, the input is the steering angle u_L, the states are the slide slip angle β and the yaw rate $\dot{\psi}$, and the measured outputs are the lateral acceleration γ_L and the yaw rate $\dot{\psi}$. Moreover, we consider the wind force as an unknown perturbation $F_\omega(t)$ and the measurement noise vector $\eta(t)$. The nonlinear model of the vehicle is give as

$$\begin{bmatrix} \dot{\beta}(t) \\ \ddot{\psi}(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_r+c_f}{mv(t)} & \frac{c_r l_r - c_f l_f}{mv^2(t)} - 1 \\ \frac{c_r l_r - c_f l_f}{I_z} & -\frac{c_r l_f^2 + c_f l_f^2}{I_z v(t)} \end{bmatrix} \begin{bmatrix} \beta(t) \\ \dot{\psi}(t) \end{bmatrix} + \begin{bmatrix} -\frac{c_f}{mv(t)} \\ \frac{c_r l_f}{I_z} \end{bmatrix} u_L(t) + \begin{bmatrix} -\frac{1}{mv(t)} \\ \frac{l_\omega}{I_z} \end{bmatrix} F_\omega(t),$$

$$y(t) = \begin{bmatrix} -\frac{c_r+c_f}{m} & \frac{c_f l_f - c_r l_r}{mv^2(t)} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta(t) \\ \dot{\psi}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{v(t)} \\ 0.2 \end{bmatrix} \eta(t),$$

where the definitions and values of all the parameters involved are displayed in Table I.

We discretize the primitive continuous-time model with a sampling time of $T = 0.001s$ by using the first-order Euler difference method and define two scheduling variables $\rho_{1,k} = \frac{1}{v}$ and $\rho_{2,k} = \frac{1}{v^2}$. Then, the nonlinear vehicle model can be transformed into a discrete-time LPV model in the form (1). The parametric matrices of the discrete-time LPV model are obtained as

$$A(\rho_k) = \begin{bmatrix} 1-0.0895\rho_{1,k} & -0.001+0.0275\rho_{2,k} \\ 0.0173 & 1-0.0903\rho_{1,k} \end{bmatrix},$$
$$B(\rho_k) = \begin{bmatrix} 0.0420\rho_{1,k} \\ 0.0299 \end{bmatrix},\ C(\rho_k) = \begin{bmatrix} -89.5358 & -27.4569\rho_{2,k} \\ 0 & 1 \end{bmatrix},$$
$$E(\rho_k) = \begin{bmatrix} 0.1009\times10^{-5}\rho_{1,k} \\ 0.2541\times10^{-6} \end{bmatrix},\ F(\rho_k) = \begin{bmatrix} \rho_{1,k} \\ 0.2 \end{bmatrix}.$$

In this example, the speed $v(t)$ varies between 20km/h and 40km/h. Since $v(t)$ is bounded, $\rho_{1,k}$ and $\rho_{2,k}$ are also bounded. This implies that a polytope bounding the vector composed of these two scheduling variables can be obtained and it has four vertices. Moreover, by using the vertices, the vehicle model can be transformed into a polytopic LPV form. Since $\rho_{1,k}$ and $\rho_{2,k}$ have a mathematical relationship, the number of the vertices of the LPV model can be reduced to three (see [13] for more details). Similarly, we consider the case that the real speed $v(t)$ cannot be obtained exactly (i.e., there always exist errors between ρ_k and its measured value $\hat{\rho}_k$ and $\tilde{\rho}_k \in \tilde{\mathcal{P}} = [(-0.002, 0.002)\ (-0.002, 0.002)]^T$). The input $u_L(k)$, the steering angle, varies between $-\frac{\pi}{12}rad$ and $\frac{\pi}{4}rad$.

In this example, we consider robust FD of the vehicle under two situations corresponding to The First Situation in Section III-B and The Second Situation in Section III-C, respectively.

For The First Situation, we assume that the unknown input $F_\omega(k)$ and measurement noise $\eta(k)$ are bounded by $W = -0.001 \oplus 0.002\mathbb{B}^1$ and $V = -0.001 \oplus 0.002\mathbb{B}^1$, respectively. Considering the decoupling conditions, we design the parametric matrices $\mathring{H} = \begin{bmatrix} 0.3779 & -0.0216 \\ -32.6685 & 1.7039 \end{bmatrix}$ and $Y_1 = \begin{bmatrix} -1.0756\times 10^{-2} & 4.0748\times 10^{7} \\ -2.0676\times 10^{-2} & 5.1472\times 10^{7} \end{bmatrix}$. Some other parametric matrices are designed as $N_k^{\hat{\rho}} = \begin{bmatrix} 0.01 & 0 \\ 0.35\hat{\rho}_{1,k}+0.15\hat{\rho}_{2,k} & 0.02 \end{bmatrix}$, $T_k^{\hat{\rho}} = \begin{bmatrix} 0.55 \\ 0.28 \end{bmatrix}$ and $K_{2,k}^{\hat{\rho}} = \begin{bmatrix} 0.76 & 0.82 \\ 0.54 & 0.27 \end{bmatrix}$. We consider that a fault in the actuator has occurred with magnitude $f = 0.1rand$, where $rand$ denotes a random variable belonging to the interval $[0, 1]$. Moreover, the initial conditions are given as

$$\hat{\Gamma}'_0 = \check{\Gamma}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0.008 & 0 & 0 & 0 \\ 0 & 0.008 & 0 & 0 \\ 0 & 0 & 0.008 & 0 \\ 0 & 0 & 0 & 0.008 \end{bmatrix} \mathbb{B}^4.$$

It is assumed that the fault occurs at time instant $k = 101$ and the FD results are shown in Figures 2 and 3. For the convenience of drawing the set-inclusion relation, we consider the interval hulls of $R_{1,k}$ and $\mathbf{R}_{1,k}$, whose first and second components are displayed in Figures 2 and 3, respectively. The green and red belts correspond to $\mathbf{R}_{1,k}$ and $R_{1,k}$, respectively. We can see that from time instant $k = 0$ to $k = 100$, the set inclusion $R_{1,k} \subseteq \mathbf{R}_{1,k}$ can always hold, which means that there is no fault occurrence. However, at $k = 101$, $R_{1,k} \nsubseteq \mathbf{R}_{1,k}$ indicates that the fault has occurred in the system.

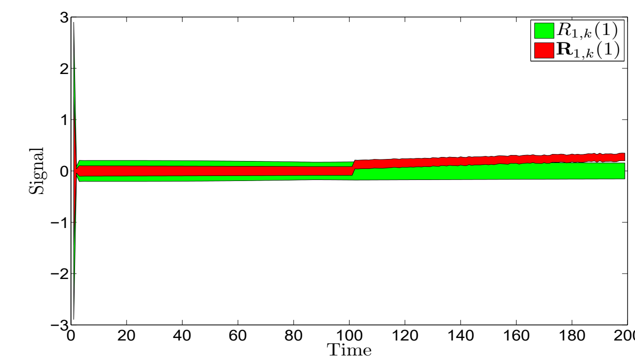

Fig. 2. Robust FD of vehicle by testing whether $R_{1,k}(1) \subseteq \mathbf{R}_{1,k}(1)$ for The First Situation

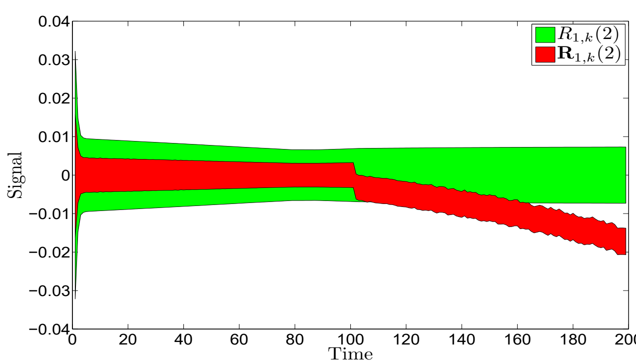

Fig. 3. Robust FD of vehicle by testing whether $R_{1,k}(2) \subseteq \mathbf{R}_{1,k}(2)$ for The First Situation

For The Second Situation, we compare the proposed robust FD method in this paper with the existing set-based bounding method in [25] from the viewpoint of FD sensitivity and aim to illustrate the advantage of the proposed robust FD method. We assume that the unknown input $F_\omega(k)$ and the measurement noise $\eta(k)$ are bounded by two known zonotopic sets $W = 0.001\mathbb{B}^1$ and $V = 1 \oplus 0.3\mathbb{B}^1$, respectively. We consider an actuator fault with magnitude $f = 0.8 + 0.1rand$. Moreover, the initial conditions are given as

$$\hat{\Gamma}'_0 = \check{\Gamma}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0.01 \end{bmatrix} \mathbb{B}^4.$$

Considering decoupling the measurement noises η_k and η_{k+1}, we set $Y_1 = \begin{bmatrix} 0 & -0.1 \\ 10^{-4} & 0 \end{bmatrix}$ and $Y_2 = \begin{bmatrix} 0 & 0.2 \\ 10^{-3} & 0 \end{bmatrix}$. By using (18) and (32), we obtain $K_{1,k}^{\hat{\rho}} = \begin{bmatrix} 0 & 0 \\ 10^{-4} & -5\times 10^{-4}\hat{\rho}_{1,k} \end{bmatrix}$ and $H_{k+1}^{\hat{\rho}} = \begin{bmatrix} 0 & 0 \\ 10^{-3} & -5\times 10^{-3}\hat{\rho}_{1,k+1} \end{bmatrix}$. Using the YALMIP toolbox, the matrices $S_{1,i}$, $G_{1,i}$, $S_{2,i}$ and $G_{2,i}$ $(i = 1, 2, 3)$ for the sufficient stability conditions in Theorem 2 are computed as

$$S_{1,1} = \begin{bmatrix} 15.2949 & 0.0117 \\ 0.0117 & 15.2290 \end{bmatrix}, \; S_{1,2} = \begin{bmatrix} 15.2949 & 0.0069 \\ 0.0069 & 15.1827 \end{bmatrix},$$
$$S_{1,3} = \begin{bmatrix} 15.2949 & 0.0064 \\ 0.0064 & 15.1754 \end{bmatrix}, \; G_{1,1} = \begin{bmatrix} 15.2702 & -0.0061 \\ 0.0178 & 15.2372 \end{bmatrix},$$
$$G_{1,2} = \begin{bmatrix} 15.2702 & -0.0077 \\ 0.0145 & 15.2139 \end{bmatrix}, \; G_{1,3} = \begin{bmatrix} 15.2702 & -0.0078 \\ 0.0142 & 15.2102 \end{bmatrix},$$
$$S_{2,1} = \begin{bmatrix} 18.9923 & 0.4324 \\ 0.4324 & 6.7811 \end{bmatrix}, \; S_{2,2} = \begin{bmatrix} 20.3414 & 0.3015 \\ 0.3015 & 7.7551 \end{bmatrix},$$
$$S_{2,3} = \begin{bmatrix} 20.3713 & 0.2292 \\ 0.2292 & 7.7404 \end{bmatrix}, \; G_{2,1} = \begin{bmatrix} 17.8914 & -0.0865 \\ 0.7643 & 7.2930 \end{bmatrix},$$
$$G_{2,2} = \begin{bmatrix} 18.3568 & -0.3777 \\ 0.4895 & 8.6667 \end{bmatrix}, \; G_{2,3} = \begin{bmatrix} 18.3693 & -0.4622 \\ 0.4619 & 8.6550 \end{bmatrix},$$

which are verified to be able to robustly stabilize the original dynamics. Additionally, the other parametric matrices of the SUIO are designed in the same way with the first situation.

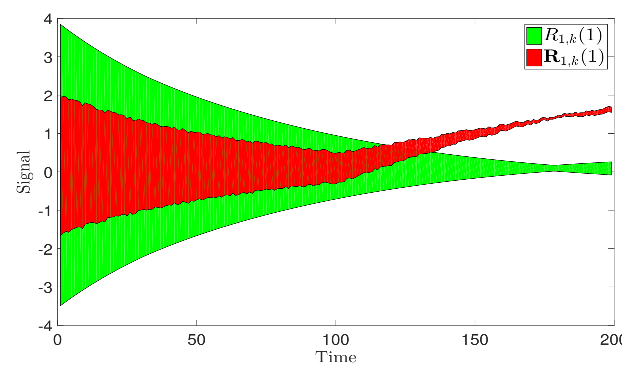

Fig. 4. Robust FD of vehicle by testing whether $R_{1,k}(1) \subseteq \mathbf{R}_{1,k}(1)$ for The Second Situation

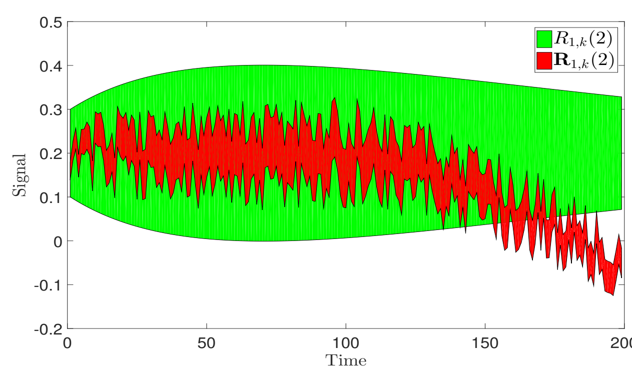

Fig. 5. Robust FD of vehicle by testing whether $R_{1,k}(2) \subseteq \mathbf{R}_{1,k}(2)$ for The Second Situation

Furthermore, for the set-based robust FD method proposed in [25], we set the initial condition as

$$\mathbb{X}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix} \mathbb{B}^2.$$

By solving the linear matrix inequality (LMI) based stability conditions in [25], the vertices of the Luenberger-structure observer gain matrix L are obtained as

$$L_1 = L_2 = L_3 = \begin{bmatrix} -0.0097 & -0.0099 \\ -0.0019 & 0.8638 \end{bmatrix}.$$

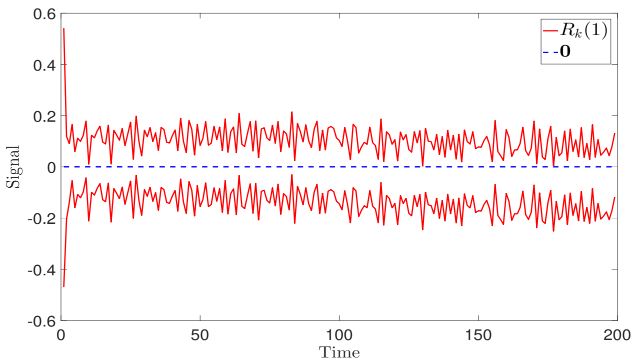

Fig. 6. Robust FD of vehicle by testing whether $\mathbf{0} \in R_k(1)$ using the method in [25]

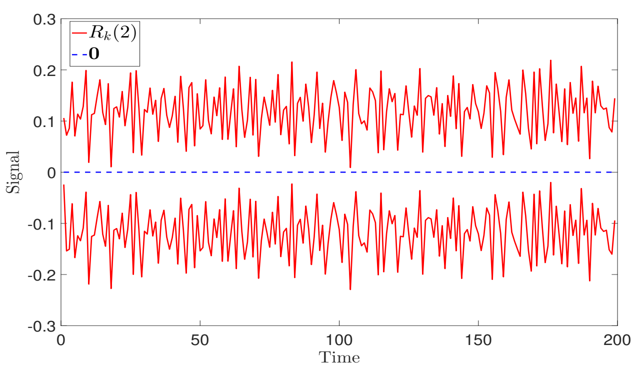

Fig. 7. Robust FD of vehicle by testing whether $\mathbf{0} \in R_k(2)$ using the method in [25]

We also assume that the fault occurs at time instant $k = 101$. The FD results in this paper are shown in Figures 4 and 5, while the FD results of the robust FD method in [25] are shown in Figures 6 and 7. Similarly, for the convenience of illustration, we only display the interval hulls of $R_{1,k}$ and $\mathbf{R}_{1,k}$ in Figures 4 and 5. In Figures 4 and 5, we can see that although there exists a time delay from the fault occurrence to the detection of the fault, the proposed method in this paper still successfully detects the fault at time instant $k = 115$ since the set inclusion relation $R_{1,k} \subseteq \mathbf{R}_{1,k}$ does not hold any longer after $k = 115$, i.e., $R_{1,115} \nsubseteq \mathbf{R}_{1,115}$. However, from Figures 6 and 7, we can find that during the whole stage of system operation, the residual sets R_k always contain the origin $\mathbf{0}$, i.e., $\mathbf{0} \in R_k$, which means that the FD method in [25] is not able to detect the occurred fault. This is because the proposed method in this paper considers the bounds of measurement errors of scheduling variables instead of the direct bounds of scheduling variables. Moreover, the proposed method in this paper can actively decouple part of the effects of measurement noises η_k on the performance of robust FD. Comparatively, the designed observer gain matrix of the FD method in [25] can only guarantee system stability and does not make similar contributions to decrease the FD conservatism. Thus, in this sense, the proposed FD method is able to achieve a higher FD performance than that of [25].

VI. Conclusions

This paper focuses on robust FD and SUIO design of LPV systems with both state and output equations scheduled by inexact scheduling variables. First, for this type of LPV systems, this paper combines the set theory with the UIO to propose a novel robust FD method. Second, the unknown input decoupling condition of SUIO for such LPV systems is given and consequently an SUIO design method is proposed for the reduced version of such LPV systems under this condition. From our best knowledge, this is the first work to work on these two problems of such LPV systems. In the future, we will consider the potential applications of the proposed ideas for fault isolation, fault estimation and fault-tolerant control.

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Appendix

Definition 2. *An m-order zonotope Z is defined as $Z = g \oplus H\mathbb{B}^m$, where g and H are called the center and segment matrix, respectively, and $\mathbb{B}^m$ is an m-dimensional unitary box whose elements are the unitary intervals $[-1,\ 1]$.*

For simplicity, a zonotope $Z = g \oplus H\mathbb{B}^m$ is denoted as $Z(g, H)$. Moreover, some properties of zonotopes taken from [1], [15] are further recalled in the following.

Property 1. *Given two zonotopes $Z_1 = Z(g_1,\ H_1)$ and $Z_2 = Z(g_2,\ H_2)$, $Z_1 \oplus Z_2 = Z(g_1 + g_2,\ [H_1\ H_2])$.*

Property 2. *Given a zonotope $Z = Z(g, H)$ and a compatible matrix K, $KZ = Z(Kg,\ KH)$.*

Property 3. *Given a family of zonotopes denoted by $X = g \oplus \mathbf{M}\mathbb{B}^m$, where $g \in \mathbb{R}^n$ is a real vector, $\mathbf{M} \in \mathbb{R}^{n\times m}$ is an interval matrix and $\mathbb{B}^m$ is an m-order unitary interval vector, a zonotope inclusion $\diamond(X)$ is defined by $\diamond(X) = g \oplus [mid(\mathbf{M})\ \ G]\mathbb{B}^{m+n}$, where the matrix G is a diagonal matrix with $G_{ii} = \sum_{j=1}^{m} \frac{diam(\mathbf{M})_{ij}}{2}$, $i = 1, 2, \cdots, n$.*

Property 4. *Given $X_{k+1} = \mathbf{A}X_k^e \oplus \mathbf{B}u_k$, where $\mathbf{A}$ and $\mathbf{B}$ are interval matrices and u_k is the input at time instant k, if X_k^e is a zonotope with the center g_k^e and segment matrix H_k^e, X_{k+1} can be bounded by a zonotope $X_{k+1}^e = g_{k+1}^e \oplus H_{k+1}^e\mathbb{B}^r$ (i.e., $X_{k+1} \subseteq X_{k+1}^e$), where $g_{k+1}^e = mid(\mathbf{A})g_k^e + mid(\mathbf{B}))u_k$, $H_{k+1}^e = [J_1 \quad J_2 \quad J_3]$, $J_1 = seg(\diamond(\mathbf{A}H_k^e))$, $J_2 = \frac{diam(\mathbf{A})}{2}g_k^e$ and $J_3 = \frac{diam(\mathbf{B})}{2}u_k$, where $seg(\cdot)$ computes the segment matrix of a zonotope.*

Property 5. *Given a zonotope $X = g \oplus G\mathbb{B}^r \subset \mathbb{R}^n$ and an integer s (with $n < s < r$), denote by $\hat{G}$ the matrix resulting from the reordering of the columns of the matrix G in decreasing Euclidean norm. $X \subseteq g \oplus [\hat{G}_T \quad Q]\mathbb{B}^s$ where $\hat{G}_T$ is obtained from the first $s - n$ columns of the matrix $\hat{G}$ and $Q \in \mathbb{R}^{n\times n}$ is a diagonal matrix whose elements satisfy $Q_{ii} = \sum_{j=s-n+1}^{r} \mid \hat{G}_{ij} \mid$, $i = 1, \ldots, n$.*

Proof of Theorem 1: *A Lyapunov function for robust stability of the dynamics (24) is defined as*

$$V(\tilde{\zeta}_k) = \tilde{\zeta}_k^T P_k^{\hat{\rho}} \tilde{\zeta}_k, \tag{66}$$

where $P_k^{\hat{\rho}} = \sum_{i=1}^{N} \lambda_{i,k}^{\hat{\rho}} P_i$ is a symmetric positive definite parameter-dependent matrix. Thus, we could further obtain

$$\Delta V(\tilde{\zeta}_k) = V(\tilde{\zeta}_{k+1}) - V(\tilde{\zeta}_k) = \begin{bmatrix} \tilde{\zeta}_k^T & \omega_k^T & \eta_{k+1}^T & \eta_k^T \end{bmatrix} \Omega \begin{bmatrix} \tilde{\zeta}_k \\ \omega_k \\ \eta_{k+1} \\ \eta_k \end{bmatrix}$$

with

$$\Omega=\begin{bmatrix}\Phi_0'^T P_{k+1}^{\hat\rho}\Phi_0'-P_k^{\hat\rho} & \Phi_0'^T P_{k+1}^{\hat\rho}\Phi_2' & \Phi_0'^T P_{k+1}^{\hat\rho}\Phi_3' & \Phi_0'^T P_{k+1}^{\hat\rho}\Phi_4'\\ * & \Phi_2'^T P_{k+1}^{\hat\rho}\Phi_2' & \Phi_2'^T P_{k+1}^{\hat\rho}\Phi_3' & \Phi_2'^T P_{k+1}^{\hat\rho}\Phi_4'\\ * & * & \Phi_3'^T P_{k+1}^{\hat\rho}\Phi_3' & \Phi_3'^T P_{k+1}^{\hat\rho}\Phi_4'\\ * & * & * & \Phi_4'^T P_{k+1}^{\hat\rho}\Phi_4'\end{bmatrix}.$$

In order to obtain a group of robust stability conditions based on matrix inequalities, we need to make a polytopic decomposition of Ω by using the polytopic decompositions of Φ_0', Φ_2', Φ_3' and Φ_4'. After analyzing the expressions of Φ_0', Φ_2', Φ_3', Φ_4', Λ_0, Λ_1, $\tilde\Lambda_4$, Λ_5 and $\tilde\Lambda_6$, we see that the polytopic decomposition of Ω is thoroughly determined by those of A_k^ρ, C_k^ρ, F_k^ρ, C_{k+1}^ρ, F_{k+1}^ρ, $E_k^{\hat\rho}$, $H_{k+1}^{\hat\rho}$, $K_{1,k}^{\hat\rho}$, $N_k^{\hat\rho}$, $P_k^{\hat\rho}$, $C_{k+1}^{\hat\rho}$, $P_{k+1}^{\hat\rho}$, $\sum_{i=1}^m E^i\tilde\rho_k^i$, $\sum_{i=1}^m F^i\tilde\rho_k^i$ and $\sum_{i=1}^m C^i\tilde\rho_{k+1}^i$. Thus, by making an one-by-one decomposition of those aforementioned scheduling-dependent variables based on their own polytopic decompositions given in Theorem 1, we finally obtain

$$\Omega=\sum_{i_0=1}^N\sum_{i_1=1}^N\sum_{j_1=1}^N\sum_{g_1=1}^N\sum_{h_1=1}^N\sum_{l_1=1}^{O_1}\sum_{t_1=1}^{O_2}\sum_{b_1=1}^N\sum_{d_1=1}^N\sum_{i_2=1}^N\sum_{j_2=1}^N\sum_{g_2=1}^N\sum_{h_2=1}^N\sum_{l_2=1}^{O_1}\sum_{t_2=1}^{O_2}\sum_{b_2=1}^N\sum_{d_2=1}^N\lambda_{i_0,k+1}^{\hat\rho}\lambda_{i_1,k+1}^{\hat\rho}\lambda_{j_1,k}^{\hat\rho}\lambda_{g_1,k}^{\rho}\lambda_{h_1,k+1}^{\rho}\times\lambda_{l_1,k+1}^{\hat\rho^h}\lambda_{t_1,k}^{\hat\rho^{k_1}}\lambda_{b_1,k+1}^{\tilde\rho}\lambda_{d_1,k}^{\tilde\rho}\lambda_{i_2,k+1}^{\hat\rho}\lambda_{j_2,k}^{\hat\rho}\lambda_{g_2,k}^{\rho}\lambda_{h_2,k+1}^{\rho}\lambda_{l_2,k+1}^{\hat\rho^h}\times\lambda_{t_2,k}^{\hat\rho^{k_1}}\lambda_{b_2,k+1}^{\tilde\rho}\lambda_{d_2,k}^{\tilde\rho}\Omega_{i_0,i_1,i_2,j_1,j_2,g_1,g_2,h_1,h_2,l_1,l_2,t_1,t_2,b_1,b_2,d_1,d_2}.$$

Thus, if there exist positive definite matrices P_{i_0} and P_{j_1} and matrices $\bar H_l$, $\bar K_{1,t}$, N_j and $K_{1,j}$ such that the matrix inequalities $\Omega_{i_0,i_1,i_2,j_1,j_2,g_1,g_2,h_1,h_2,l_1,l_2,t_1,t_2,b_1,b_2,d_1,d_2}\prec 0$ hold for $\forall\, i,\ j,\ g,\ h,\ b,\ d=1,\ 2,\cdots,\ N$, $l=1,\ 2,\cdots,\ O_1$ and $t=1,\ 2,\cdots,\ O_2$ (i.e., $\Delta V(\tilde\zeta_k)<0$), it is concluded that the designed parametric matrices can robustly stabilize the system. Thus, the proof is completed. □

Proof of Theorem 2: *From Lemma 2, the stability condition of Φ_0 can be formulated into the simultaneous stability problems of $N_k^{\hat\rho}$ and $\Lambda_0=A_k^\rho-H_{k+1}^{\hat\rho}C_{k+1}^\rho A_k^\rho-K_{1,k}^{\hat\rho}C_k^\rho$.*

Since the stability conditions of $N_k^{\hat\rho}$ have been obtained based on the results in Lemma 3, which is omitted here. Thus, this proof focuses on the stability conditions of the matrix $\Lambda_0=A_k^\rho-H_{k+1}^{\hat\rho}C_{k+1}^\rho A_k^\rho-K_{1,k}^{\hat\rho}C_k^\rho$.

For the stability of Λ_0, with the results in Lemma 3, we can first derive the following stability conditions:

$$\begin{bmatrix}S_{2,j} & *\\ G_{2,i}\Lambda_{0,i} & G_{2,i}+G_{2,i}^T-S_{2,i}\end{bmatrix}\succ 0 \tag{67}$$

with $\Lambda_{0,i}=A_i-H_{k+1}^{\hat\rho}C_{k+1}^\rho A_i-K_{1,k}^{\hat\rho}C_i$ by decomposing $A_k^\rho=\sum_{i=1}^N\lambda_{i,k}^\rho A_i$ and $C_k^\rho=\sum_{i=1}^N\lambda_{i,k}^\rho C_i$ for $i=1,2,\cdots,N$, where $S_{2,i}$ and $S_{2,j}$ are symmetric definite matrices with proper dimensions, and $G_{2,i}$ are matrices with proper dimensions. Furthermore, with the polytopic decomposition of $C_{k+1}^\rho=\sum_{h=1}^N\lambda_{h,k+1}^\rho C_h$, $H_{k+1}^{\hat\rho}=\sum_{g=1}^{O_1}\lambda_{g,k+1}^{\hat\rho^h}\bar H_g$ and $K_{1,k}^{\hat\rho}=\sum_{l=1}^{O_2}\lambda_{l,k}^{\hat\rho^{k_1}}\bar K_{1,l}$, we can have

$$\Lambda_{0,i}=\sum_{h=1}^N\sum_{g=1}^{O_1}\sum_{l=1}^{O_2}\lambda_{h,k}^\rho\lambda_{g,k+1}^{\hat\rho^h}\lambda_{l,k}^{\hat\rho^{k_1}}(A_i-\bar H_gC_hA_i-\bar K_{1,l}C_i).$$

Thus, we are able to further have

$$\begin{bmatrix}S_{2,j} & *\\ G_{2,i}\Lambda_{0,i} & G_{2,i}+G_{2,i}^T-S_{2,i}\end{bmatrix}=\sum_{h=1}^N\sum_{g=1}^{O_1}\sum_{l=1}^{O_2}\lambda_{h,k}^\rho\lambda_{g,k+1}^{\hat\rho^h}\lambda_{l,k}^{\hat\rho^{k_1}}\begin{bmatrix}S_{2,j} & *\\ G_{2,i}(A_i-\bar H_gC_hA_i-\bar K_{1,l}C_i) & G_{2,i}+G_{2,i}^T-S_{2,i}\end{bmatrix},$$

which implies that as long as (31b) holds for $\forall\ h=1,2,\cdots,N$, $g=1,\ 2,\cdots,\ O_1$ and $l=1,\ 2,\cdots,\ O_2$, the matrix $\begin{bmatrix}S_{2,j} & *\\ G_{2,i}\Lambda_{0,i} & G_{2,i}+G_{2,i}^T-S_{2,i}\end{bmatrix}$ is also positive definite. Thus, according to Lemmas 2 and 3, the dynamics (28) is stable. □

Proof of Theorem 3: *First, a set-based version of (56) can be obtained as $\Gamma_{k+1}=\Phi_0\Gamma_k\oplus\Phi_3V\oplus\Phi_4V$, where Γ_k is an unknown and uncomputable set. As a result, if $\tilde\zeta_0\in\Gamma_0$, as long as (56) is stable for all $\rho_k\in\mathcal{P}$, we can always have $\tilde\zeta_k\in\Gamma_k$ for all $k>0$. Second, we define the error vector*

$$\tilde x_k=\xi_k-\bar\xi_k=x_k-\bar\xi_k \tag{68}$$

and by substituting (52b) and (55b) into (68), we can have $\tilde x_k=\Psi_0\tilde\zeta_k$. Furthermore, it can be obtained that

$$\tilde x_k\in\Psi_0\Gamma_k. \tag{69}$$

By using the set-based observer (59), we can estimate the set of $\bar\xi_k$ in real time, i.e., $\bar\xi_k\in\hat\Xi_k$. Similar to (59), we can also consider the interval matrices $\mathbf{\Phi}_0$, $\mathbf{\Phi}_3$ and $\mathbf{\Phi}_4$ of Φ_0, Φ_3 and Φ_4 to obtain a computable bounding set $\mathbf{\Gamma}_k$ of Γ_k based on $\mathbf{\Gamma}_{k+1}=\mathbf{\Phi}_0\mathbf{\Gamma}_k\oplus\mathbf{\Phi}_3V\oplus\mathbf{\Phi}_4V$ with $\tilde\zeta_k\in\Gamma_k\subseteq\mathbf{\Gamma}_k$. Thus, with (69), we obtain $x_k\in\hat X_k=\hat\Xi_k\oplus\Psi_0\mathbf{\Gamma}_k$. □

Proof of Theorem 4: *A Lyapunov function for robust stability analysis of the dynamics (56) is defined as (66). Thus, we could further obtain*

$$\Delta V(\tilde\zeta_k)=V(\tilde\zeta_{k+1})-V(\tilde\zeta_k)=\begin{bmatrix}\tilde\zeta_k^T & \eta_{k+1}^T & \eta_k^T\end{bmatrix}\Omega\begin{bmatrix}\tilde\zeta_k\\ \eta_{k+1}\\ \eta_k\end{bmatrix},$$

with

$$\Omega=\begin{bmatrix}\Phi_0^TP_{k+1}^{\hat\rho}\Phi_0-P_k^{\hat\rho} & \Phi_0^TP_{k+1}^{\hat\rho}\Phi_3 & \Phi_0^TP_{k+1}^{\hat\rho}\Phi_4\\ * & \Phi_3^TP_{k+1}^{\hat\rho}\Phi_3 & \Phi_3^TP_{k+1}^{\hat\rho}\Phi_4\\ * & * & \Phi_4^TP_{k+1}^{\hat\rho}\Phi_4\end{bmatrix}.$$

Like Proof of Theorem 1, we need to make a polytopic decomposition of Ω to obtain a group of robust stability conditions based on matrix inequalities for (56) by using the polytopic decompositions of Φ_0, Φ_3 and Φ_4. After analyzing the expressions of Φ_0, Φ_3, Φ_4, Λ_0', Λ_1', Λ_5' and Λ_6', we see that the polytopic decomposition of Ω is thoroughly determined by those of A_k^ρ, F_k^ρ, F_{k+1}^ρ, $K_{1,k}^{\hat\rho}$, $N_k^{\hat\rho}$, $P_k^{\hat\rho}$ and $P_{k+1}^{\hat\rho}$. Thus, by making an one-by-one decomposition of those aforementioned scheduling-dependent variables in Ω based on their own polytopic decompositions given in Theorem 4, we obtain

$$\Omega=\sum_{i=1}^N\sum_{j_1=1}^N\sum_{g_1=1}^N\sum_{j_2=1}^N\sum_{g_2=1}^N\sum_{h_1=1}^N\sum_{h_2=1}^N\lambda_{i,k+1}^{\hat\rho}\lambda_{j_1,k}^{\hat\rho}\lambda_{g_1,k}^{\rho}\lambda_{j_2,k}^{\hat\rho}\times\lambda_{g_2,k}^{\rho}\lambda_{h_1,k+1}^{\rho}\lambda_{h_2,k+1}^{\rho}\Omega_{i,j_1,j_2,g_1,g_2,h_1,h_2}.$$

According to the Lyapunov stability theory, if there exist positive definite matrices P_i and P_{j_1} and matrices H, N_j

and $K_{1,j}$ such that the matrix inequalities (62) hold for $\forall i, j_1, j_2, g_1, g_2, h_1, h_2 = 1, 2, \cdots, N$ (i.e., $\Delta V(\tilde{\zeta}_k) < 0$), it is concluded that the designed parametric matrices can robustly stabilize the SUIO. Thus, the proof is completed. $\square$

Proof of Theorem 5: *According to Lemma 2, the stability problem of Φ_0 can be formulated into the simultaneous stability problems of $N_k^{\hat{\rho}}$ and $\Lambda_0' = A_k^{\rho} - HCA_k^{\rho} - K_{1,k}^{\hat{\rho}}C$.*

Thus, to assure the stability of $N_k^{\hat{\rho}}$, with the results in Lemma 3, we can have the stability conditions (65a) for $\forall\ i,\ j = 1, 2, \cdots, N$, where $S_{1,i}$ and $S_{1,j}$ are symmetric definite matrices with proper dimensions, $G_{1,i}$ are matrices with proper dimensions and $N_k^{\hat{\rho}} = \sum_{i=1}^{N} \lambda_{i,k}^{\hat{\rho}} N_i$.

Similarly, to assure the stability of Λ_0', with the results in Lemma 3, we can have the following stability conditions:

$$\begin{bmatrix} S_{2,j} & * \\ G_{2,i}\Lambda_{0,i}' & G_{2,i} + G_{2,i}^T - S_{2,i} \end{bmatrix} \succ 0 \qquad (70)$$

with $\Lambda_{0,i}' = A_k^{\rho} - HCA_k^{\rho} - K_{1,i}C$ and $K_{1,k}^{\hat{\rho}} = \sum_{i=1}^{N} \lambda_{i,k}^{\hat{\rho}} K_{1,i}$, $\forall\ i,\ j = 1, 2, \cdots, N$, where $S_{2,i}$ and $S_{2,j}$ are symmetric definite matrices with proper dimensions, and $G_{2,i}$ are matrices with proper dimensions.

Furthermore, by using the polytopic decomposition of $A_k^{\rho} = \sum_{g=1}^{N} \lambda_{g,k}^{\rho} A_g$, we can have $\Lambda_{0,i}' = \sum_{g=1}^{N} \lambda_{g,k}^{\rho}(A_g - HCA_g - K_{1,i}C)$. Thus, we can further have

$$\begin{aligned} &\begin{bmatrix} S_{2,j} & * \\ G_{2,i}\Lambda_{0,i}' & G_{2,i}+G_{2,i}^T-S_{2,i} \end{bmatrix} \\ =&\sum_{g=1}^{N} \lambda_{g,k}^{\rho} \begin{bmatrix} S_{2,j} & * \\ G_{2,i}(A_g-HCA_g-K_{1,i}C) & G_{2,i}+G_{2,i}^T-S_{2,i} \end{bmatrix}, \end{aligned}$$

which implies that if (65b) holds for $\forall\ g = 1, 2, \cdots, N$, the matrix $\begin{bmatrix} S_{2,j} & * \\ G_{2,i}\Lambda_{0,i}' & G_{2,i}+G_{2,i}^T-S_{2,i} \end{bmatrix}$ is positive definite. Thus, under Theorem 5, the dynamics (64) is stable. $\square$

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